Split Polynomials and the Sullivan Conjecture

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Contents

1 Introduction

1.1 Background and motivation

The motivation for this thesis begins with complete intersections. First, we consider a single positive integer d, which will be the degree of a homogeneous polynomial $f \in \mathbf{C}[X_0, \ldots, X_{n+1}]$. Then the vanishing locus

$$X_n(f, d) := \{ [z] \in \mathbf{C}P^{n+1} \mid f(z) = 0 \}$$

is an algebraic variety. When 0 is a regular value of f, it is a smooth complex variety embedded in $\mathbf{C}P^{n+1}$ of complex codimension 1, called a *hypersurface*.

Example 1.1.1. When $n = 1$, $X_1(f, d)$ is diffeomorphic to the oriented genus g surface $F_g \subseteq \mathbf{C}P^2$ for some g.

Now consider a finite multiset of positive integers $\underline{d} = \{d_1, \ldots, d_k\}$, and let $f_1, \ldots, f_k \in \mathbf{C}[X_0, \ldots, X_{n+k}]$ be homogeneous polynomials of degrees $d_1, \ldots, d_k$ respectively. If 0 is a regular value of each f_i and the vanishing locus

$$\begin{aligned} X_n(f_1, \ldots, f_k, \underline{d}) &:= \{ [z] \in \mathbf{C}P^{n+k} \mid f_i(z) = 0 \text{ for all } i = 1, \ldots, k \} \\ &= X_{n+k-1}(f_1, d_1) \cap \cdots \cap X_{n+k-1}(f_k, d_k) \end{aligned}$$

is the transverse intersection of $X_{n+k-1}(f_i, d_i)$, $i = 1, \ldots, k$, then $X_n(f_1, \ldots, f_k, \underline{d})$ is a smooth complex variety embedded in $\mathbf{C}P^{n+k}$ of complex codimension k. This is what we call a *complete intersection*. The diffeomorphism type of $X_n(f_1, \ldots, f_k, \underline{d})$ depends only on the multidegree $\underline{d}$, a result often attributed to Thom, but is elaborated upon in [CN23, §2.1], and so we write

$$X_n(\underline{d}) := X_n(f_1, \ldots, f_k)$$

ambiguously for its diffeomorphism type. [LW82, Theorem 8.2] provides a sort of converse to this statement.

A key point of interest in the study of complete intersections is its connection to the Sullivan Conjecture. A version of the conjecture due to Crowley and Nagy [CN23], which we state for exposition without defining all the relevant terms, is as follows.

Conjecture 1.1.2 (The Sullivan Conjecture). *Denote by $d = d_1 \cdots d_k$ the* total degree *of $X_n(\underline{d})$, and let $\chi(X_n(\underline{d}))$ be its* Euler characteristic. *Then if $n \geqslant 3$, two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic if*

1. *$d = d'$;*

2. *$\chi(X_n(\underline{d})) = \chi(X_n(\underline{d}'))$; and*

3. *The stable normal bundles of $X_n(\underline{d})$ and $X_n(\underline{d}')$ are isomorphic.*

We briefly call the three objects listed above the Sullivan data *of a complete intersection* $X_n(\underline{d})$.

For a fixed value of n, the Sullivan data depends only on certain polynomial functions of the individual degrees $d_1, \ldots, d_n$. A consequence of this conjecture is therefore a large supply of examples of complex manifolds which do not have the same complex structure, but which have the same Sullivan data by coincidence (a pigeonhole argument can be made for example), and therefore have the same underlying smooth structure.

Another K-theoretic formulation of the conjecture, also due to Crowley and Nagy [CN23] is as follows.

Conjecture 1.1.3 (The Sullivan Conjecture, K-theoretic version). *For $n \geqslant 3$, two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic if their* normal invariants *$\eta(\underline{d})$ and $\eta(\underline{d}')$ are equal.*

We speak more about *normal invariants* at the beginning of Chapter 5. At a basic level, normal invariants are constructed out of the data of fibrewise polynomial maps between line bundles over a space. In order to investigate these normal invariants, we develop the theory of *split polynomials* a model space in which we are able to observe the behaviour of the fibrewise polynomial maps in question.

1.2 Outline of results

We now provide an outline of the structure of the thesis along with an overview of what we have achieved.

The main object of study of this thesis is a topological monoid called the *split polynomials*, and our results are concerning the structure of the split polynomial space, and its associated quotient under a unitary action, called the $\mathcal{A}$*-space*.

In Chapter 2, we prove some auxiliary results which are used in later chapters. In the second half of the chapter, we define the theory of fibrewise degree-d maps between vector bundles following the work of Brumfiel and Madsen [BM76].

In Chapter 3, we give the definition of the *split polynomial space* and the $\mathcal{A}$*-space*. Our main results are concerning the structure of the $\mathcal{A}$-space, such as Theorem 3.4.6 (Relations in $\mathcal{A}(n)_{p^2}$) and Theorem 3.4.12 (Relations in $\mathcal{A}(n)_{pq}$). Based on these results, we describe a certain stratification of the $\mathcal{A}$-space depending on the commutativity of atomic split polynomial maps in certain factorisations of a general element of the split polynomial space.

In Chapter 4, we construct a model for the classifying space $(QS^0/U)_d$ for fibrewise degree-d maps of complex vector bundles as a homotopy orbit space $U \mathbin{/\mkern-6mu/} QS^0_d$. We give two different proofs: one for Theorem 4.2.2 (A classifying space for $\mathcal{F}^{\mathrm{ts}}_{d,n}$) in the unstable context, and one for Theorem 4.3.3 (A classifying space for $\mathcal{F}_d$) in the stable context.

In Chapter 5, we prove a result that the homotopy quotient $U(n+1) \mathbin{/\mkern-6mu/} SP(n)_d$ is homotopy equivalent to $\mathcal{A}(n)_d$ (Theorem 5.1.1), establishing the $\mathcal{A}$-space as a subspace of the classifying space $U \mathbin{/\mkern-6mu/} QS^0_d$. In the second half of the chapter, we compute the isomorphism type of the canonical vector bundle over the $\mathcal{A}$-space restricted to the maximal anti-diagonal. This is Theorem 5.2.3. A corollary of this is the vector bundle over the atomic $\mathcal{A}$-space, stated as Theorem 5.2.8.

In Chapter 6, we compute the cohomology of the $\mathcal{A}$-space in various degrees, including when the degree is: the square of a prime, and the product of two distinct primes.

2 Preliminaries

In this thesis, we assume that the reader is familiar with concepts of algebraic and differential topology. In this chapter, we state and prove a some results that will be used in later chapters. We also state the definition of the notion of a *fibrewise degree-d map*, which were defined by Brumfiel and Madsen [BM76] and later studied by Crowley and Nagy [CN23] in their work on the Sullivan Conjecture.

2.1 Principal bundles

We begin with a result about the induced bundle of a restriction of a principal bundles. To establish the context, we also provide definitions of principal bundles, which we take from [Hus94].

Definition 2.1.1 (G-space with free action). [Hus94, Section 4.2, Definition 2.1] Let G be a topological group. A *(right) G-space* is a space P with a right G-action. We say that G acts *freely* on P if $pg = p$ implies $g = 1_G$, i.e., only the identity of G fixes any point of P. Let P^* be the subspace of all $(p, pg) \in P \times P$, where $p \in P$ and $g \in G$. There is a function $\tau : P^* \to G$, called the *translation function*, such that $p\,\tau(p, p') = p'$ for all $(p, p') \in P^*$.

Remark 2.1.2. Of course, there is an analogous version for a left G-action, called a left G-space.

Definition 2.1.3 (G-bundle). [Hus94, Section 4.1, Definition 1.6] Let G be a topological group acting on a space P on the right. A G-bundle is a map $p : P \to X$ such that there exists a homeomorphism $f : P/G \to X$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, equal] \arrow[d] & P \arrow[d, "p"] \\ P/G \arrow[r, "f"'] & X. \end{tikzcd}$$

Definition 2.1.4 (Principal G-bundle). [Hus94, Section 4.2, Definition 2.2] A G-space P with free G-action is called *principal* if the translation function $\tau : P^* \to G$ is continuous. A *principal G-bundle* is a G-bundle $p : P \to X$, where P is a principal G-space.

A principal G-bundle is then a fibre bundle with fibre G.

Remark 2.1.5. Usually, we also assume that a principal G-bundle admits *local trivialisations*. That is, there exists an open cover $\{U_\alpha\}$ of X such that restricted to each open set U_α, there exists a G-equivariant homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ taking each fibre $p^{-1}(x)$ to $\{x\} \times G$ by a continuous group isomorphism:

$$\begin{tikzcd}
U_\alpha \times G \arrow[r, phantom, "\overset{h_\alpha}{\cong}"] \arrow[dr, "\mathrm{pr}_{U_\alpha}"'] & p^{-1}(U_\alpha) \arrow[r, hook] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "p"] \\
& U_\alpha \arrow[r, hook] & X.
\end{tikzcd}$$

[Hus94] calls such bundles *numerable*.

The following definition and proposition will be used in the context of constructing homotopy orbit spaces in the later chapters.

Definition 2.1.6 (Balanced product). [Hus94, Section 4.5] Let P be a right G-space and F be a left G-space. Then the product $P \times F$ can be made into a right G-space via the action $(p, f)g = (pg, g^{-1}f)$. The quotient $(P \times F)/G$ is denoted by $P \times_G F$, and is called the *balanced product* of P and F.

Proposition 2.1.1 (Constructing a fibre bundle from a principal G-bundle). *[Hus94, Section 4.5, Proposition 5.3] Let $p : P \to X$ be a principal G-bundle and F be a left G-space. The composition $P \times F \xrightarrow{\mathrm{pr}_P} P \xrightarrow{p} X$ factors through the balanced product as $P \times F \to P \times_G F \to X$, and we denote the resulting map $P \times_G F \to X$ by $p \times_G F$. The map $p \times_G F$ is a fibre bundle with fibre F.*

Definition 2.1.7 (Restriction of a principal bundle). [Hus94, Section 6.2, Definition 2.1] Let $P \to X$ be a principal G-bundle. Let $Q \to X$ be a principal H-bundle, where H is a closed subgroup of G. Suppose there exists an H-equivariant map $f : Q \to f(Q) \subseteq P$ which is a homeomorphism onto the closed subset $f(Q)$. Then the bundle $Q \to X$ is called a *restriction* of $P \to X$ to H.

Here is the main result of this section.

Lemma 2.1.8 (Induced bundle). *Let $p : P \to X$ be a principal G-bundle and $q : Q \to X$ be a principal H-bundle where $i : Q \hookrightarrow P$ is a closed subset. Then there is a commutative diagram*

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[d, hook] & G \arrow[d, hook] \\
Q \arrow[r, hook, "i"] \arrow[d, "q"'] & P \arrow[d, "p"'] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

and the induced principal G-bundle $q' = q \times_H G : Q \times_H G \to X$ is isomorphic to p.

Proof. Consider the map

$$\begin{aligned} f : \quad Q \underset{H}{\times} G &\longrightarrow \quad P \\ [q, g] &\longmapsto \quad i(q)g. \end{aligned}$$

There is a commutative diagram

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[d, hook] & G \arrow[r, two heads] \arrow[d, hook] & G/H \arrow[d, hook] \\
Q \arrow[r, hook, "i"] \arrow[d, "q"'] & P \arrow[r, two heads] \arrow[d, "p"'] & P/H \arrow[d, "p/H"'] \\
X \arrow[r, equal] & X \arrow[r, equal] & X, \arrow[u, bend right, "s"']
\end{tikzcd}$$

where the section $s : X \to P/H$ is given by $x \mapsto i(Q_x)$. The action of G/H is transitive on each fibre, so f is surjective. The map f is clearly injective because the action of G on P is free, and i is injective. The inverse map can be constructed as follows: For each $x \in X$, we select a point $q \in Q_x \subseteq P_x$. Then on the fibre over x, we map via

$$f_x^{-1} : P_x \longrightarrow Q_x \underset{H}{\times} G, \qquad p \longmapsto [q, \tau(q,p)].$$

This map is independent of the choice of q, for if we pick another $q' \in Q_x$, we have $q' = q\,\tau(q,q')$ and $p = q\,\tau(q,p) = q'\,\tau(q,q')^{-1}\,\tau(q,p)$ so that

$$[q, \tau(q,p)] = [q\,\tau(q,q'), \tau(q,q')^{-1}\tau(q,p)] = [q', \tau(q',p)].$$

So f is a homeomorphism. $\square$

2.2 Turning polynomial maps into maps of spheres

We now exhibit a relationship between a certain class of "well-behaved" maps $\mathbf{C}^n \to \mathbf{C}^n$ and maps of spheres $S^{2n-1} \to S^{2n-1}$. The main motivation for our definitions will be the desire to turn a non-constant polynomial map $\mathbf{C}^n \to \mathbf{C}^n$ whose preimage of $\{0\}$ is $\{0\}$ into an element of $\mathrm{Map}(S^{2n-1}, S^{2n-1})$, so this should be the prototypical example to bear in mind when reading through this section.

Let $\widehat{\mathbf{C}}^n := \mathbf{C}^n \cup \{\infty\}$ denote the one-point compactification of $\mathbf{C}^n$. The space $\widehat{\mathbf{C}}^n$ is homeomorphic to the (unreduced) suspension of S^{2n-1}, and therefore can be written as a quotient of the cylinder $S^{2n-1} \times I$. This quotient is realised by the "polar coordinates" map:

$$\begin{array}{rccc} q: & S^{2n-1} \times I & \longrightarrow & \widehat{\mathbf{C}}^n \\ & (\theta, r) & \longmapsto & r\theta/(1-r), \end{array}$$

where $S^{2n-1} \subseteq \mathbf{C}^n$ is the unit sphere, and the expression $r\theta/(1-r)$ is interpreted appropriately as giving the point at infinity when $r = 1$.

Definition 2.2.1. We define $\mathrm{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$ to be the subspace of $\mathrm{Map}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$ consisting of maps $f : \widehat{\mathbf{C}}^n \to \widehat{\mathbf{C}}^n$ satisfying:

1. The preimages $f^{-1}(0) = \{0\}$ and $f^{-1}(\infty) = \{\infty\}$; and

2. There exists a lift $\widetilde{f} : S^{2n-1} \times I \to S^{2n-1} \times I$ of f such that the following diagram commutes:

$$\begin{tikzcd} S^{2n-1} \times I \arrow[r, "\widetilde{f}"] \arrow[d, "q"'] & S^{2n-1} \times I \arrow[d, "q"] \\ \widehat{\mathbf{C}}^n \arrow[r, "f"] & \widehat{\mathbf{C}}^n. \end{tikzcd}$$

Remark 2.2.2 (Uniqueness of extensions). We remark that a priori, condition 2 implies the existence of a lift $\widetilde{f} : S^{2n-1} \times \operatorname{Int} I \to S^{2n-1} \times \operatorname{Int} I$. Then the lift in condition 2 exists if and only if $\widetilde{f}$ is *uniformly continuous* by compactness of $S^{2n-1} \times I$, in which case its extension to $S^{2n-1} \times I$ is *unique*.

On $\operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$, there is a *normalising* map

$$\begin{array}{rccc} N: & \operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n) & \longrightarrow & \operatorname{Map}(S^{2n-1}, S^{2n-1}) \\ & f & \longmapsto & \dfrac{f|_{S^{2n-1}}}{\| f|_{S^{2n-1}} \|}, \end{array}$$

and a *suspension* map

$$\begin{array}{rccc} S: & \operatorname{Map}(S^{2n-1}, S^{2n-1}) & \longrightarrow & \operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n) \\ & f & \longmapsto & Sf, \end{array}$$

where

$$\begin{array}{rccc} Sf: & \widehat{\mathbf{C}}^n & \longrightarrow & \widehat{\mathbf{C}}^n \\ & z \notin \{0,\infty\} & \longmapsto & \|z\|\, f(z/\|z\|). \end{array}$$

Indeed, $\operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$ and $\operatorname{Map}(S^{2n-1}, S^{2n-1})$ are both monoids under composition, but we remark that only S is a monoid homomorphism.

Theorem 2.2.3. *N and S are homotopy inverses.*

Remark 2.2.4. Now consider the space $\operatorname{Poly}_0(\mathbf{C}^n, \mathbf{C}^n)$ of polynomial maps $p : \mathbf{C}^n \to \mathbf{C}^n$ such that $p^{-1}(0) = \{0\}$. So p is non-constant and can be extended to a map $\widehat{\mathbf{C}}^\infty \to \widehat{\mathbf{C}}^\infty$. This defines an inclusion $\operatorname{Poly}_0(\mathbf{C}^n, \mathbf{C}^n) \hookrightarrow \operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$. What Theorem 2.2.3 provides us is a way to go between the algebra of polynomial maps and the well-studied topological space of maps of spheres.

Proof of Theorem 2.2.3. Clearly $NS = \mathrm{id}$. We construct a homotopy of SN to the identity.

Each $f \in \operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$ is the quotient of a lift $\widetilde{f} : S^{2n-1} \times I \to S^{2n-1} \times I$. This lift is unique by Remark 2.2.2 Writing $\widetilde{f} = \theta \times r$, where $\theta : S^{2n-1} \times I \to S^{2n-1}$ and $r : S^{2n-1} \times I \to I$ are the two components of $\widetilde{f}$ in the product $S^{2n-1} \times I$, there is a well-defined "polar coordinates" map

$$\begin{array}{rccc} P: & \operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n) & \longrightarrow & \operatorname{Map}(S^{2n-1} \times I, S^{2n-1}) \times \operatorname{Map}(S^{2n-1} \times I, I) \\ & f & \longmapsto & (\theta, r). \end{array}$$

Due to our definition of $\operatorname{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n)$ condition 1 actually forces the image of P to be $\operatorname{Map}(S^{2n-1} \times I, S^{2n-1}) \times \operatorname{Map}_{0,1}(S^{2n-1} \times I, I)$, where $\operatorname{Map}_{0,1}(S^{2n-1} \times I, I) \subseteq \operatorname{Map}(S^{2n-1} \times I, I)$ denotes the subspace of maps $g : S^{2n-1} \times I \to I$ such that $g^{-1}(0) = S^{2n-1} \times \{0\}$ and $g^{-1}(1) = S^{2n-1} \times \{1\}$. Now, $P : \mathrm{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n) \to \mathrm{Map}(S^{2n-1} \times I, S^{2n-1}) \times \mathrm{Map}_{0,1}(S^{2n-1} \times I, I)$ is a homeomorphism.

The space $\mathrm{Map}_{0,1}(S^{2n-1} \times I, I)$ is contractible because I contractible: we can always homotope a map $g \in \mathrm{Map}_{0,1}(S^{2n-1} \times I, I)$ to the projection $\mathrm{pr}_I : S^{2n-1} \times I \to I$, which is certainly an element of $\mathrm{Map}_{0,1}(S^{2n-1} \times I, I)$. On the other hand, the space $\mathrm{Map}(S^{2n-1} \times I, S^{2n-1})$ is homeomorphic to the free path space $\mathrm{Map}(S^{2n-1}, S^{2n-1})^I$, which deformation retracts onto the subspace of constant maps homeomorphic to $\mathrm{Map}(S^{2n-1}, S^{2n-1})$ by "compressing" each path $\gamma : I \to \mathrm{Map}(S^{2n-1}, S^{2n-1})$ to the constant map $\gamma|_{1/2} : \{1/2\} \to \mathrm{Map}(S^{2n-1}, S^{2n-1})$. So we have a homotopy equivalence

$$\mathrm{Map}(S^{2n-1} \times I, S^{2n-1}) \times \mathrm{Map}_{0,1}(S^{2n-1} \times I, I) \simeq \mathrm{Map}(S^{2n-1}, S^{2n-1}),$$

which is realised as a deformation retraction onto the subspace $\mathrm{Map}(S^{2n-1} \times \{1/2\}, S^{2n-1}) \times \{\mathrm{pr}_I\} \cong \mathrm{Map}(S^{2n-1}, S^{2n-1})$. Denote this deformation retraction by r_t.

Now observe the following factorisation of N through $\mathrm{Map}(S^{2n-1} \times I, S^{2n-1}) \times \mathrm{Map}(S^{2n-1} \times I, I)$:

$$\begin{tikzcd}
\mathrm{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n) \arrow[rr, "N"] \arrow[dr, "P"', "\cong"] & & \mathrm{Map}(S^{2n-1}, S^{2n-1}) \\
& \mathrm{Map}(S^{2n-1} \times I, S^{2n-1}) \times \mathrm{Map}_{0,1}(S^{2n-1} \times I, I). \arrow[ur, "r_1"'] &
\end{tikzcd}$$

Going the other way, S is obtained via the factorisation

$$\begin{tikzcd}
\mathrm{Map}_{0,\infty}(\widehat{\mathbf{C}}^n, \widehat{\mathbf{C}}^n) & & \mathrm{Map}(S^{2n-1}, S^{2n-1}) \arrow[ll, "S"'] \arrow[dl, hook, "i"] \\
& \mathrm{Map}(S^{2n-1} \times I, S^{2n-1}) \times \mathrm{Map}_{0,1}(S^{2n-1} \times I, I), \arrow[ul, "P^{-1}", "\cong"'] &
\end{tikzcd}$$

where the inclusion i is by mapping $\mathrm{Map}(S^{2n-1}, S^{2n-1})$ to the image of the deformation retract $\mathrm{Map}(S^{2n-1} \times \{1/2\}, S^{2n-1}) \times \{\mathrm{pr}_I\} \cong \mathrm{Map}(S^{2n-1}, S^{2n-1})$. Thus, the deformation retraction r_t obtains us a homotopy of SN back to the identity. $\square$

2.3 Fibrewise degree-d maps between vector bundles

The theory of this section follows what Brumfiel and Madsen called *f-maps* in their work [BM76, §4]. We will instead work in the category of complex vector bundles, and so we will have a well-defined notion of degree of maps between complex vector bundles with respect to their preferred orientation.

Definition 2.3.1 (Fibrewise degree-d map). Let $E^n, F^n \to X$ be complex vector bundles over a connected space X of rank n, and let $S(E^n), S(F^n) \to X$ denote their sphere bundles. A *fibrewise degree-d map $f : S(E^n) \to S(F^n)$* is a fibre preserving map

$$\begin{tikzcd}
S(E^n) \arrow[rr, "f"] \arrow[dr] & & S(F^n) \arrow[dl] \\
& X &
\end{tikzcd}$$

which is of degree d on each fibre, i.e., $f_x : S(E^n)_x \to S(F^n)_x$ has degree d for each $x \in X$.

2.3.1 New fibrewise degree-d maps from old

The *join* of two spaces X and Y is the set of all formal convex combinations of points in X and Y

$$X * Y = \{ t_1 x + t_2 y \mid x \in X,\ y \in Y,\ t_1 + t_2 = 1,\ t_1, t_2 \geqslant 0 \},$$

and is topologised as a quotient of $X \times Y \times I$. Given two maps $f : X \to Z$ and $g : Y \to W$, we can define an induced map between the joins $X * Y \to Z * W$ by

$$\begin{array}{rccc} f * g : & X * Y & \longrightarrow & Z * W \\ & t_1 x + t_2 y & \longmapsto & t_1 f(x) + t_2 g(y). \end{array}$$

Where S^0 is the discrete two point space in $\mathbf{R}$, its nth iterated join $(S^0)^{*n}$ is consists of formal convex combinations of $2n$ points in the axes of $\mathbf{R}^n$. By radial projection outward from the origin, we have a homeomorphism

$$\underbrace{S^0 * \cdots * S^0}_{n \text{ times}} \cong S^{n-1}.$$

Hence, by associativity of the join, $S^{m-1} * S^{n-1} \cong S^{m+n-1}$.

Definition 2.3.2 (Direct sums). For two maps of spheres $f_i : S^{m-1} \to S^{m-1}$ with degree d_i, $i = 0, 1$, the induced map on the join $f_0 * f_1 : S^{m+n-1} \to S^{m+n-1}$ has degree $d_0 d_1$. Repeating this construction fibrewise, we can take the fibrewise join of fibrewise degree-d_i maps $f_i : S(E_i) \to S(F_i)$, $i = 0, 1$, resulting in a fibrewise degree-$d_0 d_1$ map $f_0 \oplus f_1 : S(E_0 \oplus E_1) \to S(F_0 \oplus F_1)$ between the *direct sums* of the vector bundles, i.e.

$$(f_0 \oplus f_1)_x := f_{0x} * f_{1x} : S(E_0 \oplus E_1)_x \longrightarrow S(F_0 \oplus F_1)_x$$

on each fibre over $x \in X$. We have a commutative diagram

$$\begin{tikzcd} S(E_1 \oplus E_2) \arrow[rr, "f_1 \oplus f_2"] \arrow[dr] & & S(F_1 \oplus F_2) \arrow[dl] \\ & X. & \end{tikzcd}$$

Definition 2.3.3 (Pullbacks). Given a fibrewise degree-d map $f : S(E) \to S(F)$ between sphere bundles $S(E), S(F) \to X$, and a map $g : Y \to X$, we can form the *pullback* $g^* f : S(g^* E) \to S(g^* F)$, defined on each fibre by

$$(g^* f)_y := f_{g(y)} : S(g^* E)_y \longrightarrow S(g^* F)_y$$

for all $y \in Y$. The pullback is again a fibrewise degree-d map. Hence, there is a commutative diagram

$$\begin{tikzcd}
S(g^*E) \arrow[rr, "g^*f"] \arrow[dr] \arrow[ddr] & & S(g^*F) \arrow[dr] \arrow[ddl] & \\
& S(E) \arrow[rr, "f"] \arrow[ddr] & & S(F) \arrow[ddl] \\
& Y \arrow[dr, "g"'] & & \\
& & X. &
\end{tikzcd}$$

2.3.2 Isomorphisms and homotopies of fibrewise degree-d maps

We begin by defining 3 operations that relate fibrewise degree-d maps: isomorphism, homotopy, and stable isomorphism.

Definition 2.3.4 (Isomorphism of fibrewise degree-d maps). Let $f_i : S(E_i^n) \to S(F_i^n)$, $i = 0, 1$, be two fibrewise degree-d maps over the same base space X. An *isomorphism* between f_0 and f_1 is a $U(n)$-bundle isomorphisms $g : E_0^n \to E_1^n$ and $h : F_0^n \to F_1^n$ such that the following diagram commutes:

$$\begin{tikzcd}
S(E_0^n) \arrow[rr, "f_0"] \arrow[dr, "g", "\cong"'] \arrow[ddr] & & S(F_0^n) \arrow[dr, "h", "\cong"'] \arrow[ddl] & \\
& S(E_1^n) \arrow[rr, "f_1"] \arrow[ddr] & & S(F_1^n) \arrow[ddl] \\
& X \arrow[dr, equal] & & \\
& & X. &
\end{tikzcd}$$

Definition 2.3.5 (Homotopy of fibrewise degree-d maps). Let $f_i : S(E_i^n) \to S(F_i^n)$, $i = 0, 1$, be two fibrewise degree-d maps over the same base space X. A *homotopy* between f_0 and f_1 is a fibrewise degree-d map $f : S(E^n) \to S(F^n)$, where $E^n, F^n \to X \times I$ are oriented vector bundles over $X \times I$, such that the restrictions of f to $X \times \{0\}$ and $X \times \{1\}$ are equal to f_0 and f_1 respectively. That is to say, we have the two pullback squares

$$\begin{tikzcd}
(S(E_0^n) \xrightarrow{f_0} S(F_0^n)) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (S(E^n) \xrightarrow{f} S(F^n)) \arrow[d] & (S(E_0^n) \xrightarrow{f_1} S(F_0^n)) \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
X \times \{0\} \arrow[r, hook] & X \times I & X \times \{1\}. \arrow[l, hook']
\end{tikzcd}$$

Definition 2.3.6 (Stable isomorphism of fibrewise degree-d maps). Let $f : S(E) \to S(F)$ be a fibrewise degree-d map over a connected space X. For $\alpha : G \to G'$ a $U(n)$-bundle isomorphism of vector bundles $G, G' \to X$, we say that f and $f \oplus \alpha : S(E \oplus G) \to S(F \oplus G')$ are *stably isomorphic*.

We now define the following equivalence relations on the class of fibrewise degree-d maps over a connected space X.

Definition 2.3.7 ((Unstable) homotopy equivalence). Let $f_i : S(E_i^n) \to S(F_i^n)$, $i = 0, 1$, be two fibrewise degree-d maps over the same base space X. We say that f_0 and f_1 are *(unstably) homotopy equivalent*, or *homotopic*, if they are related by the equivalence relation generated by the two operations:

1. isomorphism (see Definition 2.3.4); and
2. homotopy (see Definition 2.3.5).

We denote this equivalence relation by $\simeq$.

Remark 2.3.8. It can be shown that the definition of unstable homotopy equivalence of fibrewise degree-d maps $f_i : S(E_i^n) \to S(F_i^n)$, $i = 0, 1$ is equivalent to asking for a *single* homotopy $f : S(E^n) \to S(F^n)$, where $E^n, F^n \to X \times I$ are oriented vector bundles over $X \times I$, such that the restrictions of f to $X \times \{0\}$ and $X \times \{1\}$ are *isomorphic* to f_0 and f_1 respectively. We will not need to use this equivalence in this thesis.

Definition 2.3.9 (Stable homotopy equivalence). Let $f_i : S(E_i^n) \to S(F_i^n)$, $i = 0, 1$, be two fibrewise degree-d maps over the same base space X. We say that f_0 and f_1 are *stably homotopy equivalent*, or *stably homotopic*, if they are related by the equivalence relation generated by the three operations:

1. isomorphism (see Definition 2.3.4);
2. homotopy (see Definition 2.3.5); and
3. stable isomorphism (see Definition 2.3.6).

We denote this equivalence relation by $\simeq_s$.

Remark 2.3.10. It can be also be shown that the definition of stable homotopy equivalence of fibrewise degree-d maps is equivalent to only allowing a *single* step of stabilisation. We will not need to use this equivalence in this thesis.

2.3.3 Trivialising either the source or target

All vector bundles $E \to X$ over compact Hausdorff X have an inverse bundle, i.e., another bundle $E^\perp \to X$ such that $E \oplus E^\perp \to X$ is isomorphic to a trivial bundle by [Hat17, Proposition 1.3]. Thus, we can assume that either the source bundle or target bundle in a fibrewise degree-d map $f : S(E) \to S(F)$ is trivial through one of the following operations:

- **Trivialising the source:** We direct sum on the inverse of E:

$$(f : S(E) \to S(F)) \simeq_s (f \oplus \mathrm{id}_{S(E^\perp)} : S(E \oplus E^\perp) \to S(F \oplus E^\perp)),$$

  where the source bundle $E \oplus E^\perp$ is now a trivial bundle.

- **Trivialising the target:** We direct sum on the inverse of F:

$$(f : S(E) \to S(F)) \simeq_s (f \oplus \mathrm{id}_{S(F^\perp)} : S(E \oplus F^\perp) \to S(F \oplus F^\perp)),$$

  where the target bundle $F \oplus F^\perp$ is now a trivial bundle.

Hence, every fibrewise degree-d map is stably isomorphic to either a fibrewise degree-d map of the form

$$\begin{tikzcd} S(\underline{\mathbf{C}}^n) \arrow[rr, "f^{\mathrm{ts}}"] \arrow[dr] & & S(F) \arrow[dl] \\ & X, & \end{tikzcd}$$

or a fibrewise degree-d map of the form

$$\begin{tikzcd} S(E) \arrow[rr, "f^{\mathrm{tt}}"] \arrow[dr] & & S(\underline{\mathbf{C}}^n) \arrow[dl] \\ & X, & \end{tikzcd}$$

where $\underline{\mathbf{C}}^n$ denotes the rank n trivial bundle over X.

2.4 A fact about tensor powers of line bundles

Here, we state and prove the following lemma, which we invoke later in Section 5.2.

Lemma 2.4.1. *Let $\gamma \to X$ be a complex line bundle and let $S(\gamma) \to X$ be the associated principal S^1-bundle. Then the map $S(\gamma) \to S(\gamma^{\otimes d})$ is precisely the d-fold power map $z \mapsto z^d$ on each fibre.*

Proof. To see this, we work on a local trivialisation. Over an open $U \subseteq X$ where X is trivial, there is an isomorphism $\gamma|_U \cong U \times \mathbf{C}$. By definition of the tensor product on vector bundles, the map $\gamma \to \gamma^{\otimes d}$ of vector bundles is given locally on U by the map $U \times \mathbf{C} \to U \times \mathbf{C}^{\otimes d}$, $z \mapsto z^d$. □

3 Split polynomials and the $\mathcal{A}$-space

In this chapter, we define the notion of a *split polynomial*, which was introduced by C. Nagy in the work for his PhD. The definition of a split polynomial aims to model the tautological fibrewise map $\gamma \to \gamma^{\otimes d}$ for γ a line bundle. We provide some exposition on the structure of the split polynomials and the related $\mathcal{A}$*-space*.

Notation 3.0.1. For a positive integer d, we denote the $(2n+1)$-dimensional *lens space* by L_d^{2n+1}. It is the lens space constructed as a quotient of S^{2n+1} by the diagonal $\mathbf{Z}_d$-action generated by the map

$$(z_0, \dots, z_n) \longmapsto (e^{2\pi i/d} z_0, \dots, e^{2\pi i/d} z_n).$$

3.1 Split polynomials

For this section, we let n denote a non-negative integer.

Definition 3.1.1 (Atomic split polynomial). Let $\mathbf{C}^{n+1}$ be equipped with the standard inner product. An *atomic split polynomial* is a polynomial map of the form

$$\begin{array}{rccc} (v, d): & \mathbf{C}^{n+1} & \longrightarrow & \mathbf{C}^{n+1} \\ & z & \longmapsto & \langle z, v\rangle^d v + (z - \langle z, v\rangle v) \end{array}$$

for $v \in S^{2n+1}$ and $d \in \mathbf{Z}_{>0}$. We abuse notation and denote such an atomic split polynomial by the pair (v, d). When the degree can be inferred, we also take the liberty to elide d and simply denote an atomic split polynomial by the vector v.

Here is a more concrete way of viewing the definition of an atomic split polynomial. First, extend v to an ordered orthonormal basis $\beta(v) = (v, b_1, \dots, b_n)$ of $\mathbf{C}^{n+1}$. Now, elements of $\mathbf{C}^{n+1}$ can be expressed in coordinates with respect to this basis via

$$\begin{pmatrix} z_0 & z_1 & \cdots & z_n \end{pmatrix}_{\beta(v)} := z_0 v + z_1 b_1 + \cdots + z_n b_n.$$

The action of (v, d) is then the dth power map in 0th coordinate:

$$(v, d) \cdot \begin{pmatrix} z_0 & z_1 & \cdots & z_n \end{pmatrix}_{\beta(v)} = \begin{pmatrix} z_0^d & z_1 & \cdots & z_n \end{pmatrix}_{\beta(v)}.$$

Definition 3.1.2 (Atomic split polynomial space). Consider the space of polynomial maps formed by taking the atomic split polynomials and composing with unitary maps on both the domain and codomain. We denote this space

$$SP(n)^{\mathrm{at}} := \{\, A \circ (v, d) \circ B \mid A, B \in U(n+1),\ v \in S^{2n+1},\ d \in \mathbf{Z}_{>0} \,\},$$

and call it the *space of atomic split polynomials*. We identify $SP(n)^{\mathrm{at}}$ as a subspace of $\mathrm{Map}(\mathbf{C}^{n+1}, \mathbf{C}^{n+1})$, and give it the subspace topology.

Definition 3.1.3 (Split polynomial space). The *(general) split polynomial space* $(SP(n), \circ)$ is the submonoid of $\mathrm{Map}(\mathbf{C}^{n+1}, \mathbf{C}^{n+1})$ under composition generated by the atomic split polynomials and unitary maps. We usually denote $(SP(n), \circ)$ by just $SP(n)$, eliding the monoid operation. The split polynomials $SP(n)$ have a tautological monoid action on $\mathbf{C}^{n+1}$.

Remark 3.1.4. We mention that the split polynomials may be defined in a coordinate-free way on any (finite-dimensional) complex inner product space $(V, \langle , \rangle)$, where now, we use the unitary group $U(V)$ of inner product preserving linear maps instead of $U(n+1)$. For convenience, we will work with $\mathbf{C}^{n+1}$ throughout this thesis, but corresponding results will hold in the more general setting.

Relations in $SP(n)$. The split polynomials satisfy the following relations: for $A, B \in U(n+1)$, $v, w \in S^{2n+1}$, and $d, e \in \mathbf{Z}_{>0}$, we have

1. $A \circ B = AB$.
2. $I = 1_{SP(n)}$, where $I \in U(n+1)$ is the identity matrix, and $(v, 1) = 1_{SP(n)}$.
3. $(v, d) \circ (v, e) = (v, de)$.
4. $(v, d) \circ (w, e) = (w, e) \circ (v, d)$ for all $v \perp w$.
5. $A \circ (v, d) = (Av, d) \circ A$.
6. $(\lambda v, d) = A_v^{\lambda^{1-d}} \circ (v, d)$ for $\lambda \in S^1$, where $A_v^c \in U(n+1)$ for a constant $c \in S^1$ is the unitary map given by $A_v^c(x) = c\langle x, v\rangle v + (x - \langle x, v\rangle v)$.

Thus, one can also define the split polynomial space as the abstract monoid $(SP^{\mathrm{abs}}(n), \circ)$ generated by the symbols (v, d) for every $v \in S^{2n+1}$, $d \in \mathbf{Z}_{>0}$, and A for every $A \in U(n+1)$ subject to the above 6 relations. By fiat, there is a surjective monoid homomorphism $SP^{\mathrm{abs}}(n) \twoheadrightarrow SP(n)$.

Conjecture 3.1.5 (Equivalence of $SP^{\mathrm{abs}}(n)$ and $SP(n)$). *The surjective monoid homomorphism $SP^{\mathrm{abs}}(n) \to SP(n)$ is a monoid isomorphism.*

For this thesis, we will assume this conjecture and identify the two constructions of $SP(n)$. In the following chapters, we will prove specific cases of the above conjecture: Theorem 3.4.6 (Relations in $\mathcal{A}(n)_{p^2}$) and Theorem 3.4.12 (Relations in $\mathcal{A}(n)_{pq}$).

Given a word $f = w_1 w_2 w_3 \cdots w_k \in SP(n)$, we can use relation 5 above to bring all unitary maps $w_i \in U(n+1)$ to the left and atomic split polynomial maps to the right of the word. Combining unitary maps with relation 1, we see that any $f \in SP(n)$ admits a factorisation of the form

$$f = A \circ (v_1, d_1) \circ \cdots \circ (v_k, d_{k'}).$$

where $A \in U(n+1)$ and $v_1, \ldots, v_{k'} \in S^{2n+1}$, $d_1, \ldots, d_{k'} \in \mathbf{Z}_{>0}$.

Definition 3.1.6 (Normal form). We call a factorisation of a split polynomial as shown above a *normal form* of the split polynomial. Note that the normal form is not unique. For example, using relation 6 above, the equality $(v, d) = A_v^{\lambda^{d-1}} \circ (\lambda v, d)$ holds for all $\lambda \in S^1$.

3.2 The structure of split polynomials

In this section, we analyse some of the monoid structure of the split polynomials. In particular, we would like to answer the following two questions:

1. When are two split polynomials equal?

2. When do two split polynomials commute?

We mainly focus on the atomic case for simplicity.

We will use the following key idea for our analysis: each element $f \in SP(n)$ is a polynomial map $f : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$, and so there is an associated *Jacobian determinant* map $\det Df : \mathbf{C}^{n+1} \to \mathbf{C}$. Given two split polynomials $f, g \in SP(n)$, it is therefore a necessary condition that the functions $\det Df, \det Dg : \mathbf{C}^{n+1} \to \mathbf{C}$ are equal in order for f and g to be equal.

3.2.1 When are two split polynomials equal?

Proposition 3.2.1 (Equality of atomic split polynomials). *An atomic split polynomial* (v, d) *with* $d \neq 1$ *depends only on the equivalence class* $[v] \in L^{2n+1}_{d-1}$*, i.e.,* $(v, d) = (v', d)$ *if and only if* $[v] = [v']$ *in* L^{2n+1}_{d-1}.

Proof. We calculate the form of $\det Df$ when $f = (v, d)$, an atomic split polynomial. Let $v \in S^{2n+1}$ and $d \in \mathbf{Z}_{>0}$. Then for $z \in \mathbf{C}^{n+1}$, we obtain the formula

$$\det D(v, d)_z = d\langle z, v\rangle^{d-1}.$$

Hence, given $(v, d), (w, e) \in SP(n)$ with $d \neq 1$, $e \neq 1$, the equality $(v, d) = (w, e)$ implies $d\langle z, v\rangle^{d-1} = e\langle z, w\rangle^{e-1}$ must hold for all $z \in \mathbf{C}^{n+1}$.

Case 1. If $v \nparallel w$, w admits an orthogonal decomposition

$$w = w_\parallel + w_\perp, \quad \text{where} \quad w_\parallel \in \mathbf{C}v,\ w_\perp \in (\mathbf{C}v)^\perp \setminus \{0\}.$$

Therefore evaluating both determinants at $z = w_\perp$, we find that

$$d\langle w_\perp, v\rangle^{d-1} = 0 \neq e\langle w_\perp, w\rangle^{e-1} = e\|w_\perp\|^{2(e-1)}.$$

So $(v, d) \neq (w, e)$.

Case 2. Now if $v = \lambda w$ for some $\lambda \in S^1$, then

$$\det D(v, d)_z = d\langle z, v\rangle^{d-1} = \lambda^{1-d} d\langle z, w\rangle^{d-1}.$$

Therefore evaluating both determinants at $z = w$ yields

$$\det D(v, d)_w = d\langle w, v\rangle^{d-1} = \lambda^{1-d} d, \quad \det D(w, e)_w = e\langle w, w\rangle^{e-1} = e.$$

Setting these equal forces $d = e$ and $\lambda^{d-1} = 1$. It is now easily seen that $(\lambda w, d) = (w, d)$ when $\lambda^{d-1} = 1$. □

3.2.2 When do two split polynomials commute?

To detect when two split polynomials commute using the Jacobian determinant, we recall the chain rule of multivariable calculus:

$$\det D(f \circ g)_z = (\det Df_{g(z)})(\det Dg_z).$$

Proposition 3.2.2 (Commutativity of atomic split polynomials). *Two atomic split polynomials (v, d) and (w, e) with $d, e \neq 1$ commute if and only if one of the following holds:*

1. $v \perp w$; *or*
2. $v = \lambda w$ *for some* $\lambda \in S^1$ *with* $\lambda^{(d-1)(e-1)} = 1$.

Proof. Given $(v, d), (w, e) \in SP(n)$ with $d \neq 1$ and $e \neq 1$, the equality $(v, d) \circ (w, e) = (w, e) \circ (v, d)$ implies

$$\langle (w, e)(z), v\rangle^{d-1}\langle z, w\rangle^{e-1} = \langle (v, d)(z), v\rangle^{e-1}\langle z, v\rangle^{d-1}$$

must hold for all $z \in \mathbf{C}^{n+1}$. Evaluating at $z \in (\mathbf{C}v)^\perp$, we are forced to have

$$\langle (w, e)(z), v\rangle^{d-1}\langle z, w\rangle^{e-1} = 0.$$

So either $\langle (w, e)(z), v\rangle = 0$ for all $z \in (\mathbf{C}v)^\perp$, or $w \in \mathbf{C}v$.

Case 1. First assume the former. Then

$$\begin{aligned}\langle (w, e)(z), v\rangle &= \langle z, w\rangle^e\langle w, v\rangle + \langle z, v\rangle - \langle z, w\rangle\langle w, v\rangle \\ &= \langle z, w\rangle^e\langle w, v\rangle - \langle z, w\rangle\langle w, v\rangle = 0 \quad \text{for all} \quad z \in (\mathbf{C}v)^\perp.\end{aligned}$$

Certainly this is satisfied when $w \perp v$, in which case $(v, d) \circ (w, e) = (w, e) \circ (v, d)$ is true.

Case 2. When $w \not\perp v$, we instead require that $\langle z, w\rangle^{e-1} = 1$ for all $z \in (\mathbf{C}v)^\perp \setminus (\mathbf{C}w)^\perp$. This is not possible unless $(\mathbf{C}v)^\perp \subseteq (\mathbf{C}w)^\perp$, i.e., if $v \parallel w$, and so we must have $v = \lambda w$ for some $\lambda \in S^1$. In this case, we explicitly check commutativity: taking $cw \in \mathbf{C}^{n+1}$, we have

$$\begin{aligned}(\lambda w, d) \circ (w, e)(cw) &= (\lambda w, d)(c^e\lambda^{-1}\lambda w) = c^{de}\lambda^{-d+1}w, \\ (w, e) \circ (\lambda w, d)(c\lambda^{-1}\lambda w) &= (w, e)(c^d\lambda^{-d+1}w) = c^{de}\lambda^{-de+e}w.\end{aligned}$$

So equality holds only if $\lambda^{(d-1)(e-1)} = 1$. □

3.2.3 Unitary actions on the split polynomials

By construction, $SP(n)$ has both a left and right $U(n+1)$-action given by pre- and post-composition respectively:

$$\begin{array}{ccc} U(n+1) \times SP(n) & \longrightarrow & SP(n) \\ (A, f) & \longmapsto & A \circ f, \end{array} \quad \text{and} \quad \begin{array}{ccc} SP(n) \times U(n+1) & \longrightarrow & SP(n) \\ (f, A) & \longmapsto & f \circ A. \end{array}$$

The left action is free because each $f \in SP(n)$ is a non-constant polynomial map $\mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$, and therefore surjective, and the unitary action on the codomain $\mathbf{C}^{n+1}$ of f is free. The right action, however, is not free. An example of this is seen even in the case $n = 0$, where we have $(\lambda z)^d = z^d$ as long as $\lambda \in U(1) = S^1$ is a dth root of unity.

3.3 The $\mathcal{A}$-space

This section will deal with the quotient of the split polynomials under the unitary action defined in the previous section. As it will turn out, this quotient will reveal much of the structure of the split polynomial space.

Definition 3.3.1 ($\mathcal{A}$-space). We write $\mathcal{A}(n) := U(n+1)\backslash SP(n)$, called the *$\mathcal{A}$-space*, for the quotient of $SP(n)$ under its *left* $U(n+1)$-action.

By modding out by the left unitary action, we are left with the "split polynomial part" of the normal form factorisation.

Definition 3.3.2 (Atomic $\mathcal{A}$-space). By definition, the atomic split polynomial space $SP(n)^{\text{at}}$ is a stable subspace under the $U(n+1)$-actions. We define the quotient $\mathcal{A}(n)^{\text{at}} := U(n+1)\backslash SP(n)^{\text{at}}$ to be the *atomic $\mathcal{A}$-space*.

The atomic $\mathcal{A}$-space consists of the split polynomials which admit a normal form factorisation consisting of only a single atomic split polynomial (and possibly unitary maps).

Remark 3.3.3. We remark that we call the quotient space the "$\mathcal{A}$-space" following the work of C. Nagy during his PhD. The choice of name apparently does not have any particular meaning.

An important property of $\mathcal{A}(n)$ is that for an equivalence class $[f] \in \mathcal{A}(n)$, each $f' \in [f]$ has the same set of critical points, i.e., $\det Df'_z = 0$ if and only if $\det Df_z = 0$ for all $z \in \mathbf{C}^{n+1}$. So the set of critical points

$$Z[f] = \{\, z \in \mathbf{C}^{n+1} \mid Df_z \text{ is not surjective} \,\}$$

is an *invariant* of the equivalence class $[f]$ and there is a well-defined map

$$\begin{array}{rccc} Z: & \mathcal{A}(n) & \longrightarrow & \{\text{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array} \tag{3.1}$$

taking an equivalence class to its set of critical points. A few questions which arise now include:

1. Is Z injective, i.e., is an equivalence class in $\mathcal{A}(n)$ uniquely identified by its set of critical points?
2. How do the relations in $SP(n)$ descend to $\mathcal{A}(n)$, and are there any new relations?

3.3.1 Decomposition by degree

Each smooth map $f : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ has a degree which is an integer defined to be the (finite) sum [Mil65, §5]

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign} \det Df_x,$$

where $y \in \mathbf{C}^{n+1}$ is a regular value of f. The degree map $\deg : C^\infty(\mathbf{C}^{n+1}, \mathbf{C}^{n+1}) \to \mathbf{Z}$ is locally constant, and hence its restriction to $SP(n)$, which we continue to denote by deg, decomposes the split polynomial space into its degree-d components:

$$SP(n) = \bigsqcup_{d \in \mathbf{Z}_{>0}} SP(n)_d, \quad \text{where} \quad SP(n)_d := \deg^{-1}(d) = \{\, f \in SP(n) \mid f \text{ has degree } d \,\}.$$

We remark that polynomials and unitary maps have positive degree, and hence we only have degree-d components for $d > 0$. Furthermore, because unitary maps have degree 1, each degree-d component is stable under both the left and right $U(n+1)$-actions. So there is a corresponding decomposition of the $\mathcal{A}$-space

$$\mathcal{A}(n) = \bigsqcup_{d \in \mathbf{Z}_{>0}} \mathcal{A}(n)_d, \quad \text{where} \quad \mathcal{A}(n)_d := U(n+1) \backslash SP(n)_d.$$

Notation 3.3.4 (Atomic spaces). We write $SP(n)_d^{\mathrm{at}}$ and $\mathcal{A}(n)_d^{\mathrm{at}}$ for the degree-d components of the atomic split polynomial space and atomic $\mathcal{A}$-space respectively.

3.4 Models for the $\mathcal{A}$-space

The prime factorisation of d constrains the possible ways in which a map $f \in SP(n)_d$ with degree d can factorise into atomic split polynomials. In this section, we will provide some results describing the structure of the $\mathcal{A}$-space based on the primes that appear in the factorisation of d.

Let the prime factorisation of d be

$$d = p_1 \cdots p_k$$

for primes $p_1, \dots, p_k$ (not necessarily distinct). Because degree is multiplicative under composition, the map f must admit a factorisation into the normal form

$$f = A \circ (v_1, p_{i_1}) \circ \cdots \circ (v_k, p_{i_k})$$

for $A \in U(n+1)$, $v_1, \dots, v_k \in S^{2n+1}$, and $(i_1, \dots, i_k)$ is some permutation of $(1, \dots, k)$ giving the ordering of the prime degrees in the factorisation. Therefore in $\mathcal{A}(n)_d$, $[f] = [(v_1, p_{i_1}) \circ \cdots \circ (v_k, p_{i_k})]$.

3.4.1 The case of d arbitrary

When d is arbitrary, we state the following conjectures, for which we do not yet have proofs.

Conjecture 3.4.1 (Injectivity of Z). *Let d be a positive integer. The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_d} : & \mathcal{A}(n)_d & \longrightarrow & \{\text{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_d$ to its set of critical points is injective.

Proposition 3.4.1 (Relations in $\mathcal{A}(n)_d$). *Let d be a product of primes $p_1 \cdots p_k$. In the $\mathcal{A}$-space of degree d, the following relations are satisfied for all $v_1, \dots, v_k \in S^{2n+1}$, $\lambda \in S^1$:*

1. $\big[(v_1, p_{i_1}) \circ \cdots \circ (v_{j-1}, p_{i_{j-1}}) \circ (\lambda v_j, p_{i_j}) \circ (v_{j+1}, p_{i_{j+1}}) \circ \cdots \circ (v_k, p_{i_k})\big]$
$= \big[(A_{v_j}^{\lambda^{p_{i_j}-1}} v_1, p_{i_1}) \circ \cdots \circ (A_{v_j}^{\lambda^{p_{i_j}-1}} v_{j-1}, p_{i_{j-1}}) \circ (v_j, p_{i_j}) \circ (v_{j+1}, p_{i_{j+1}}) \circ \cdots \circ (v_k, p_{i_k})\big].$

2. *If either $v_j \parallel v_{j+1}$ or $v_j \perp v_{j+1}$ then*

$$\begin{aligned}&[(v_1, p_{i_1}) \circ \cdots \circ (v_j, p_{i_j}) \circ (v_{j+1}, p_{i_{j+1}}) \circ \cdots \circ (v_k, p_{i_k})] \\ &\qquad = [(v_1, p_{i_1}) \circ \cdots \circ (v_{j+1}, p_{i_{j+1}}) \circ (v_j, p_{i_j}) \circ \cdots \circ (v_k, p_{i_k})].\end{aligned}$$

Proof. That these relations hold is an exercise in applying Relations in $SP(n)$ inductively. The details are omitted. □

Conjecture 3.4.2. *The relations described in Proposition 3.4.1 are the* only *relations in* $\mathcal{A}(n)_d$.

The above conjectures may be a subject of future study.

3.4.2 The case of $d = p^k$, p prime.

When the degree d is a power of a prime p^k, the normal form factorisation of $f \in SP(n)$ becomes

$$f = A \circ (v_1, p) \circ \cdots \circ (v_k, p)$$

for $A \in U(n+1)$ and $v_1, \ldots, v_k \in S^{2n+1}$, and correspondingly $[f] = [(v_1, p) \circ \cdots \circ (v_k, p)]$ in $\mathcal{A}(n)_{p^k}$. Importantly, the degrees of the atomic split polynomials in the factorisation of f are all equal to the prime p. Thus, for brevity, we may elide the p in the factorisation of an element of $\mathcal{A}(n)_{p^k}$ without ambiguity in the ordering of the primes.

We restate Conjecture 3.4.1 (Injectivity of Z) and Conjecture 3.4.2 specialised to the case when $d = p^k$.

Conjecture 3.4.3 (Injectivity of Z for $d = p^k$). *The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_{p^k}} : & \mathcal{A}(n)_{p^k} & \longrightarrow & \{\textit{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_{p^k}$ to its set of critical points is injective.

Conjecture 3.4.4 (Relations in $\mathcal{A}(n)_{p^k}$). *In the $\mathcal{A}$-space of degree p^k, the following relations are satisfied for all $v_1, \ldots, v_k \in S^{2n+1}$, $\lambda \in S^1$:*

1. $[v_1 \circ \cdots \circ v_{i-1} \circ \lambda v_i \circ v_{i+1} \circ \cdots \circ v_k] = [A_{v_i}^{\lambda^{p-1}} v_1 \circ \cdots \circ A_{v_i}^{\lambda^{p-1}} v_{i-1} \circ v_i \circ v_{i+1} \circ \cdots \circ v_k]$.
2. $[v_1 \circ \cdots \circ v_i \circ v_{i+1} \circ \cdots \circ v_k] = [v_1 \circ \cdots \circ v_{i+1} \circ v_i \circ \cdots \circ v_k]$ *if either* $v_i \parallel v_{i+1}$ *or* $v_i \perp v_{i+1}$.

Furthermore, these are the only *relations in* $\mathcal{A}(n)_{p^k}$.

For this thesis, we will assume the truth of these conjectures. However, we have positive results in the special case $k = 2$.

Theorem 3.4.5 (Injectivity of Z for $d = p^2$). *The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_{p^2}} : & \mathcal{A}(n)_{p^2} & \longrightarrow & \{\textit{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_{p^2}$ to its set of critical points is injective.

Theorem 3.4.6 (Relations in $\mathcal{A}(n)_{p^2}$). *In the $\mathcal{A}$-space of degree p^2, the following relations are satisfied for all $v, w \in S^{2n+1}$, $\lambda \in S^1$:*

1. $[\lambda v \circ w] = [v \circ w]$ *and* $[v \circ \lambda w] = [A_w^{\lambda^{p-1}} v \circ w]$.
2. $[v \circ w] = [w \circ v]$ *if either* $v \parallel w$ *or* $v \perp w$.

Furthermore, these are the only *relations in* $\mathcal{A}(n)_{p^2}$.

We omit these proofs here, and provide them in Appendix A.

We now aim to build a model for $\mathcal{A}(n)_{p^k}$ assuming Conjecture 3.4.4 (Relations in $\mathcal{A}(n)_{p^k}$). Consider the following iterated *twisted balanced product*

$$\widetilde{\mathcal{A}}(n)_{p^k} := (\cdots((\mathbf{C}P^n \underset{S^1}{\widetilde{\times}} \overbrace{L^{2n+1}_{p-1}) \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1}) \underset{S^1}{\widetilde{\times}} \cdots) \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1}}^{k-1 \text{ times}}, \tag{3.2}$$

defined inductively by the following process:

- $\widetilde{\mathcal{A}}(n)_p$ is a copy of $\mathbf{C}P^n$.
- $\widetilde{\mathcal{A}}(n)_{p^2} = \widetilde{\mathcal{A}}(n)_p \widetilde{\times}_{S_1} L^{2n+1}_{p-1}$ is the quotient of the product $\widetilde{\mathcal{A}}(n)_p \times L^{2n+1}_{p-1} = \mathbf{C}P^n \times L^{2n+1}_{p-1}$ under the S^1 action

$$\begin{array}{ccc} S^1 \times (\mathbf{C}P^n \times L^{2n+1}_{p-1}) & \longrightarrow & \mathbf{C}P^n \times L^{2n+1}_{p-1} \\ (\mu^{p-1}, \quad ([v],[w])) & \longmapsto & ([A_w^{\mu^{1-p}} v], [\mu w]). \end{array}$$

- In general for $k \geqslant 2$, $\widetilde{\mathcal{A}}(n)_{p^k} = \widetilde{\mathcal{A}}(n)_{p^{k-1}} \widetilde{\times}_{S_1} L^{2n+1}_{p-1}$ is the quotient of the product $\widetilde{\mathcal{A}}(n)_{p^{k-1}} \times L^{2n+1}_{p-1}$ under the S^1 action

$$\begin{array}{ccc} S^1 \times (\widetilde{\mathcal{A}}(n)_{p^{k-1}} \times L^{2n+1}_{p-1}) & \longrightarrow & \widetilde{\mathcal{A}}(n)_{p^{k-1}} \times L^{2n+1}_{p-1} \\ (\mu^{p-1}, \ ([v_1,\ldots,v_{k-1}],[v_k])) & \longmapsto & ([A_{v_k}^{\mu^{1-p}} v_1, \ldots, A_{v_k}^{\mu^{1-p}} v_{k-1}], [\mu v_k]). \end{array}$$

Of course, $\widetilde{\times}_{S^1}$ is *not* associative. The twisted balanced product imposes relation 1 of Conjecture 3.4.4 (Relations in $\mathcal{A}(n)_{p^k}$). To impose relation 2, we further define the equivalence relation

$$[v_1,\ldots,v_i,v_{i+1},\ldots,v_k] \sim_{p^k} [v_1,\ldots,v_{i+1},v_i,\ldots,v_k] \quad \text{if and only if} \quad v_i \parallel v_{i+1} \text{ or } v_i \perp v_{i+1}.$$

Corollary 3.4.1. *The map* $\widetilde{\mathcal{A}}(n)_{p^k}/\sim_{p^k} \to \mathcal{A}(n)_{p^k}$ *sending an equivalence class* $[v_1,\ldots,v_k] \in \widetilde{\mathcal{A}}(n)_{p^k}/\sim_{p^k}$ *to the equivalence class* $[v_1 \circ \cdots \circ v_k] \in \mathcal{A}(n)_{p^k}$ *is a homeomorphism.*

3.4.3 The case of $d = p$, p prime

We briefly consider the atomic case when d is a prime. On $\widetilde{\mathcal{A}}(n)_p$, the equivalence relation $\sim_p$ is the identity relation. Therefore $\mathcal{A}(n)_p \cong \widetilde{\mathcal{A}}(n)_p \cong \mathbf{C}P^n$ is just complex projective space.

Remark 3.4.7 (The atomic $\mathcal{A}$-space as complex projective space). In fact, the above result holds true because $\mathcal{A}(n)_p$ is an *atomic* $\mathcal{A}$-space. In general for arbitrary d, we have still have a homeomorphism $\mathcal{A}(n)_d^{\mathrm{at}} \cong \mathbf{C}P^n$.

3.4.4 The case of $d = p^2$, p prime

We now consider the case when d is the square of a prime more closely. From Corollary 3.4.1, $\mathcal{A}(n)_{p^2}$ is identified with the quotient

$$\mathcal{A}(n)_{p^2} \cong \frac{\mathbf{C}P^n \,\widetilde{\times}_{S^1}\, L^{2n+1}_{p-1}}{\sim_{p^2}}, \quad \text{where} \quad [v,w] \sim_{p^2} [w,v] \text{ if and only if } v \perp w \text{ or } v \parallel w.$$

We remark that $[v,w] = [w,v]$ for $v \parallel w$ in $\mathbf{C}P^n \,\widetilde{\times}_{S^1}\, L^{2n+1}_{p-1}$ already. Now, the definition of $\sim_{p^2}$ suggests that we should consider the following two distinguished subspaces of $\mathcal{A}(n)_{p^2}$.

Definition 3.4.8 (Diagonal and anti-diagonal). Define the two subspaces of $\mathbf{C}P^n \,\widetilde{\times}_{S^1}\, L^{2n+1}_{p-1}$

$$\begin{aligned} \Delta \; &= \{\, [v,w] \in \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1} \mid v \parallel w \,\}, \quad \text{and} \\ \Delta^- &= \{\, [v,w] \in \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1} \mid v \perp w \,\}. \end{aligned}$$

We call their images in the quotient $\mathcal{A}(n)_{p^2}$ the *diagonal* and the *anti-diagonal* of $\mathcal{A}(n)_{p^2}$ respectively.

Observe the following properties:

- Δ is homeomorphic the diagonal $\Delta_{\mathbf{C}P^n} \subseteq \mathbf{C}P^n \times \mathbf{C}P^n$. To see this, consider the subspace of $\mathbf{C}P^n \times L^{2n+1}_{p-1}$ consisting of pairs $([v],[w])$ with $v \parallel w$. The orbit of $([v],[w])$ under the S^1-action consists of elements of the form

  $$\mu^{p-1} \cdot ([v],[w]) = ([A_w^{\mu^{1-p}} v], [\mu w]) = ([\mu^{1-p} v], [\mu w]) = ([v], [\mu w]).$$

  So the S^1-action restricted to this subspace is trivial on the $\mathbf{C}P^n$ factor.

  Because $\sim_{p^2}$ is the identity relation when restricted to Δ, Δ is homeomorphic to its image under the map $\mathbf{C}P^n \,\widetilde{\times}_{S^1}\, L^{2n+1}_{p-1} \to \mathcal{A}(n)_{p^2}$. We also denote the the image Δ and freely identify the two spaces.

- Similarly, Δ^- is homeomorphic *anti-diagonal* of $\mathbf{C}P^n$, defined

  $$\Delta^-_{\mathbf{C}P^n} = \{\, ([v],[w]) \in \mathbf{C}P^n \times \mathbf{C}P^n \mid v \perp w \,\}.$$

  This is because the S^1-action restricted to the subspace of $\mathbf{C}P^n \times L^{2n+1}_{p-1}$ consisting of pairs $([v],[w])$ with $v \perp w$ is again trivial on the $\mathbf{C}P^n$ factor.

  The equivalence relation $\sim_{p^2}$ restricted to Δ^- is precisely the orbit relation under $\mathbf{Z}_2$-action swapping the two factors, defined $t \cdot [v,w] = [w,v]$ for $t \in \mathbf{Z}_2$ the generator. The quotient $\Delta^-/\mathbf{Z}_2$ is a subspace of $\mathcal{A}(n)_{p^2}$.

The two subspaces Δ and $\Delta^-/\mathbf{Z}_2$ define a *stratification* of the $\mathcal{A}$-space consisting of the following strata:

1. The top stratum $\mathcal{A}(n)_{p^2} \setminus (\Delta \sqcup \Delta^-/\mathbf{Z}_2)$, consisting of equivalence classes of split polynomials which are the composition of two atomic split polynomials (v,p) and (w,p) in *generic position*, i.e., $v \nparallel w$ and $v \not\perp w$. These are the atomic split polynomials that *do not commute* by Proposition 3.2.2.

2. The bottom stratum $\Delta \sqcup \Delta^-/\mathbf{Z}_2$, consisting of equivalence classes of split polynomials which are the composition of two atomic split polynomials that *do commute* by Proposition 3.2.2.

3.4.5 The case of d is a product of distinct primes

If d is a product of distinct primes $p_1 \cdots p_k$, the stratification of $\mathcal{A}(n)_d$ has a simple description based on the permutation of the primes in the normal form factorisation.

We again restate Conjecture 3.4.1 (Injectivity of Z) and Conjecture 3.4.2 specialised to the case when $d = p_1 \cdots p_k$.

Conjecture 3.4.9 (Injectivity of Z for d a product of distinct primes). *Let d be a product of distinct primes. The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_d}: & \mathcal{A}(n)_d & \longrightarrow & \{\textit{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_d$ to its set of critical points is injective.

Conjecture 3.4.10 (Relations in $\mathcal{A}(n)_d$). *Let d be a product of distinct primes $p_1 \cdots p_k$. In the $\mathcal{A}$-space of degree d, the following relations are satisfied for all $v_1, \ldots, v_k \in S^{2n+1}$, $\lambda \in S^1$:*

1. $\big[(v_1, p_{i_1}) \circ \cdots \circ (v_{j-1}, p_{i_{j-1}}) \circ (\lambda v_j, p_{i_j}) \circ (v_{j+1}, p_{i_{j+1}}) \circ \cdots \circ (v_k, p_{i_k})\big]$
$= \big[(A_{v_j}^{\lambda^{p_{i_j}-1}} v_1, p_{i_1}) \circ \cdots \circ (A_{v_j}^{\lambda^{p_{i_j}-1}} v_{j-1}, p_{i_{j-1}}) \circ (v_j, p_{i_j}) \circ (v_{j+1}, p_{i_{j+1}}) \circ \cdots \circ (v_k, p_{i_k})\big].$
2. *If either $v_j \parallel v_{j+1}$ or $v_j \perp v_{j+1}$ then*

$$\begin{aligned} &[(v_1, p_{i_1}) \circ \cdots \circ (v_j, p_{i_j}) \circ (v_{j+1}, p_{i_{j+1}}) \circ \cdots \circ (v_k, p_{i_k})] \\ &\qquad = [(v_1, p_{i_1}) \circ \cdots \circ (v_{j+1}, p_{i_{j+1}}) \circ (v_j, p_{i_j}) \circ \cdots \circ (v_k, p_{i_k})]. \end{aligned}$$

Furthermore, these are the only *relations in $\mathcal{A}(n)_d$.*

For this thesis, we will assume the truth of these conjectures. However, we again have positive results in the special case $k = 2$.

Theorem 3.4.11 (Injectivity of Z for $d = pq$). *Let p and q be distinct primes. The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_{pq}}: & \mathcal{A}(n)_{pq} & \longrightarrow & \{\textit{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_{pq}$ to its set of critical points is injective.

Theorem 3.4.12 (Relations in $\mathcal{A}(n)_{pq}$). *Let p and q be distinct primes, and let $\{d, e\} = \{p, q\}$. In the $\mathcal{A}$-space of degree pq, the following relations are satisfied for all $v, w \in S^{2n+1}$, $\lambda \in S^1$:*

1. $[(\lambda v, d) \circ (w, e)] = [(v, d) \circ (w, e)]$ *and* $[(v, d) \circ (\lambda w, e)] = [(A_w^{\lambda^{e-1}} v, d) \circ (w, e)]$.
2. $[(v, d) \circ (w, e)] = [(w, e) \circ (v, d)]$ *if either $v \parallel w$ or $v \perp w$.*

Furthermore, these are the only *relations in $\mathcal{A}(n)_{pq}$.*

The proofs are provided in Appendix A.

Consider now the following iterated *twisted balanced product*

$$\widetilde{\mathcal{A}}(n)_{p_{i_1}, \ldots, p_{i_k}} := (\cdots((\mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p_{i_1}-1}) \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p_{i_2}-1}) \underset{S^1}{\widetilde{\times}} \cdots) \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p_{i_k}-1},$$

which is defined analogously to the construction in Section 3.4.2 (c.f. equation (3.2)) for each permutation of the primes $(p_1, \ldots, p_k)$. The $\mathcal{A}$-space now has the following description: it is a quotient of the disjoint union

$$\widetilde{\mathcal{A}}(n)_d := \coprod_{\substack{\text{permutations}\\ i_1,\ldots,i_k}} \widetilde{\mathcal{A}}(n)_{p_{i_1},\ldots,p_{i_k}}$$

under the equivalence relation $\sim_d$ which imposes relation 2 of Conjecture 3.4.10 (Relations in $\mathcal{A}(n)_d$).

3.4.6 The case of $d = pq$, p, q distinct primes.

We again consider the specific case when d is the product of two distinct primes more closely. The $\mathcal{A}(n)_{pq}$ space is constructed by taking the disjoint union of the two spaces

$$\widetilde{\mathcal{A}}(n)_{p,q} = \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{q-1} \quad \text{and} \quad \widetilde{\mathcal{A}}(n)_{q,p} = \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1},$$

and quotienting out by the equivalence relation generated by the relations

$$\widetilde{\mathcal{A}}(n)_{d,e} \ni [v,w] \sim_{pq} [w,v] \in \widetilde{\mathcal{A}}(n)_{e,d} \quad \text{if} \quad v \perp w \text{ or } v \parallel w$$

for $\{d,e\} = \{p,q\}$. The quotient map $\widetilde{\mathcal{A}}(n)_{pq} \twoheadrightarrow \widetilde{\mathcal{A}}(n)_{pq}/{\sim_{pq}} = \mathcal{A}(n)_{pq}$ restricts to homeomorphisms on the subspaces $\widetilde{\mathcal{A}}(n)_{p,q}$ and $\widetilde{\mathcal{A}}(n)_{q,p}$; we will write $\mathcal{A}(n)_{p,q}$ and $\mathcal{A}(n)_{q,p}$ for their homeomorphic images respectively. Like for the $d = p^2$ case, we consider the following two distinguished subspaces.

Definition 3.4.13 (Diagonal and anti-diagonal). Define the subspaces

$$\Delta_{p,q} = \{[v,w] \in \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{q-1} \mid v \parallel w\}, \ \Delta_{q,p} = \{[w,v] \in \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1} \mid v \parallel w\}, \text{ and}$$
$$\Delta^-_{p,q} = \{[v,w] \in \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{q-1} \mid v \perp w\}, \Delta^-_{q,p} = \{[w,v] \in \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1} \mid v \perp w\}.$$

In the quotient $\mathcal{A}(n)_{pq}$, the images of $\Delta_{p,q}$ and $\Delta_{q,p}$ are identified through $\sim_{pq}$, and the same is true for $\Delta^-_{p,q}$ and $\Delta^-_{q,p}$. We denote their common images by Δ and Δ^- respectively, which we call the *diagonal* and *anti-diagonal* of $\mathcal{A}(n)_{pq}$.

Following the discussion of Section 3.4.4:

- Δ is homeomorphic to the diagonal $\Delta_{\mathbf{C}P^n} \subseteq \mathbf{C}P^n \times \mathbf{C}P^n$.
- Δ^- is homeomorphic to the anti-diagonal $\Delta^-_{\mathbf{C}P^n} \subseteq \mathbf{C}P^n \times \mathbf{C}P^n$, this time with no additional quotient by a $\mathbf{Z}_2$-action.

The subspaces Δ and Δ^- define a stratification of $\mathcal{A}(n)_{pq}$, but we additionally have a stratification of the top stratum given by the two subspaces $\mathcal{A}(n)_{p,q}$ and $\mathcal{A}(n)_{q,p}$. These subspaces specify the ordering of prime degrees:

- If $[f] \in \mathcal{A}(n)_{p,q}$ then f has a normal form $A \circ (v,p) \circ (w,q)$ where the degree-p map is to the left of the degree-q map.
- If $[f] \in \mathcal{A}(n)_{q,p}$, then f has a normal form $A \circ (w,q) \circ (v,p)$ where the degree-q map is to the left of the degree-p map.

Remark 3.4.14. The notion of the *diagonal* and *anti-diagonal* generalises to when d is a product of more than two primes, but the analysis quickly becomes much more complicated. So we leave these cases for future investigation.

3.5 Stabilisation

For each positive integer n, there is an inclusion map $i_n : SP(n) \hookrightarrow SP(n+1)$ induced by the inclusion $\mathbf{C}^{n+1} \hookrightarrow \mathbf{C}^{n+2}$. More precisely, writing $\mathbf{C}^{n+2} = \mathbf{C}^{n+1} \times \mathbf{C}$, we define for $f \in SP(n)$

$$i_n(f) := f \times \mathrm{id}_{\mathbf{C}} : \mathbf{C}^{n+1} \times \mathbf{C} \longrightarrow \mathbf{C}^{n+1} \times \mathbf{C},$$

giving rise to the commutative diagram

$$\begin{tikzcd} \mathbf{C}^{n+1} \times \mathbf{C} \arrow[r, "i_n(f)=f\times \mathrm{id}_{\mathbf{C}}"] & \mathbf{C}^{n+1} \times \mathbf{C} \\ \mathbf{C}^{n+1} \arrow[u, hook] \arrow[r, "f"] & \mathbf{C}^{n+1}. \arrow[u, hook] \end{tikzcd}$$

These inclusions allow us to have a well-defined notion of stabilisation for split polynomials, which we will briefly explore in this section.

Definition 3.5.1 (Stable split polynomial space). We define the *stable split polynomial space* to be the direct limit with respect to the family of inclusions $i_n : SP(n) \hookrightarrow SP(n+1)$, which we denote

$$SP := \varinjlim_n SP(n).$$

Correspondingly, we denote the degree-d component of SP by SP_d.

The space SP has left and right unitary actions by the *stable* unitary group U induced by the left and right $U(n+1)$-actions on $SP(n)$ defined in Section 3.2.3. The left action remains free, while the right action is not free. So there is a corresponding stable $\mathcal{A}$-space defined as follows.

Definition 3.5.2 (Stable $\mathcal{A}$-space). The *stable $\mathcal{A}$-space* is quotient $\mathcal{A} := U\backslash SP$, and correspondingly we denote the degree-d component by $\mathcal{A}_d$.

Alternatively, we can see that the inclusions $i_n : SP(n) \hookrightarrow SP(n+1)$ descend to the quotients $\bar{i}_n : \mathcal{A}(n) \hookrightarrow \mathcal{A}(n+1)$, and therefore we have a natural homeomorphism

$$\varinjlim_n \mathcal{A}(n) \cong \mathcal{A}.$$

Stabilisation of $\mathcal{A}(n)_{p^2}$ and $\mathcal{A}(n)_{pq}$. In terms of our models for $\mathcal{A}(n)_{p^2}$ and $\mathcal{A}(n)_{pq}$, there are inclusions $\mathbf{C}P^n \hookrightarrow \mathbf{C}P^{n+1}$, $L^{2n+1}_{p-1} \hookrightarrow L^{2n+3}_{p-1}$ and $L^{2n+1}_{q-1} \hookrightarrow L^{2n+3}_{q-1}$ induced by $\mathbf{C}^{n+1} \hookrightarrow \mathbf{C}^{n+2}$. Therefore, we also have natural homeomorphisms

$$\varinjlim_n \frac{\mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{p-1}}{\sim_{p^2}} \cong \mathcal{A}_{p^2} \quad \text{and} \quad \varinjlim_n \frac{(\mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{p-1}) \amalg (\mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{q-1})}{\sim_{pq}} \cong \mathcal{A}_{pq}.$$

4 The classifying spaces $(QS^0/U)_d$ and $U\backslash\backslash QS^0_d$

In the theory of fibrewise degree-d maps developed by Brumfiel and Madsen, they identify a classifying space for fibrewise degree-d maps between oriented real vector bundles up to $O(n)$-bundle isomorphisms, denoted by $(QS^0/O)_d$ [BM76, §4]. In this chapter, we will study the complex version, which we denote appropriately by $(QS^0/U)_d$, and construct a model for this classifying space explicitly as the homotopy orbits $U\backslash\backslash QS^0_d$.

4.1 Constructing the universal bundle

In this section, we begin by constructing the finite-dimensional version of the homotopy orbit space $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$. We then establish the existence of a universal fibrewise degree-d map over $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$, which will allow us to prove that $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ is a model for the classifying space for fibrewise degree-d maps between bundles of finite rank n.

Recall that $EU(n) \to BU(n)$ is the universal bundle over the classifying space $BU(n)$ for principal $U(n)$-bundles. A model for $BU(n)$ is the infinite-dimensional Grassmannian $G_n(\mathbf{C}^\infty)$ [Hat17, Theorem 1.16]. This is the model we will work with in this thesis. The total space $EU(n)$ is then frame bundle of the associated tautological vector bundle $VU(n) \to BU(n)$. We will denote points of $EU(n)$ as pairs (l, L), where

- $l \subseteq \mathbf{C}^\infty$ is an n-plane, equipped with an inner product which is the restriction of the canonical inner product on $\mathbf{C}^\infty$; and
- $L : \mathbf{C}^n \to l$ is an orthonormal frame for l, i.e., a unitary map $\mathbf{C}^n \to l$ with respect to the inner products on $\mathbf{C}^n$ and l.

The orthonormal frame L defines coordinates on l via the assignment

$$(z_1, \dots, z_n) \overset{L}{\longmapsto} z_1 v_1 + \cdots + z_n v_n,$$

where $v_1, \dots, v_n$ is the orthonormal basis of l given by $v_i = L(e_i)$, $i = 1, \dots, n$, for $e_1, \dots, e_n$ the standard basis of $\mathbf{C}^n$.

Definition 4.1.1. Let n and d be a positive integers. The *homotopy orbit space* $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ is defined to be the balanced product

$$U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d := EU(n) \underset{U(n)}{\times} \mathrm{Map}(S^{2n-1}, S^{2n-1})_d,$$

where $EU(n)$ has its usual right $U(n)$-action and $\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ has a left $U(n)$-action given by pre-composition:

$$\begin{array}{ccc} U(n) \times \mathrm{Map}(S^{2n-1}, S^{2n-1})_d & \longrightarrow & \mathrm{Map}(S^{2n-1}, S^{2n-1})_d \\ (g, f) & \longmapsto & g \circ f. \end{array}$$

The homotopy orbit space comes equipped with a projection map to the classifying space for principal $U(n)$-bundles, which we denote by $p_n : U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \to BU(n)$.

For this section, we use following notation for points of $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$: each point $[(l, L), f] \in U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ is denoted as an equivalence class, where

- $f : S^{2n-1} \to S^{2n-1}$ is a degree d map of $(2n-1)$-spheres; and
- (l, L) is a point of $EU(n)$ consisting of an n-plane l and an orthonormal frame L.
- The equivalence class $[(l, L), f]$ as set has the description

$$[(l, L), f] = \{\, ((l, L \circ g), g^{-1} \circ f) \mid g \in U(n) \,\}.$$

Remark 4.1.2 (A point about notation). It is more typical to denote the homotopy orbit space by $\mathrm{Map}(S^{2n-1}, S^{2n-1})_d /\!\!/ U(n)$. However, in our case, we have both a left and right $U(n)$-action on $\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ given by pre- and post-composition. In order to emphasise which action we are quotienting by, we choose to use the more unconventional notation of $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$.

4.1.1 The canonical vector bundle over $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$

Recall that $VU(n) \to BU(n)$ is the canonical vector bundle over $BU(n)$. Pulling back $VU(n)$ along the projection map $p_n : U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \to BU(n)$, we obtain the canonical vector bundle over $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$

$$\begin{tikzcd} p_n^* VU(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & VU(n) \arrow[d] \\ U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \arrow[r, "p_n"'] & BU(n). \end{tikzcd}$$

As a topological space, the total space of the pullback bundle $p_n^* VU(n)$ is the subspace of the product $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \times VU(n)$ given by

$$p_n^* VU(n) = \{\, ([(l, L), f], v) \mid [(l, L), f] \in U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d, v \in l \,\}.$$

4.1.2 Universal fibrewise degree-d map over $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$

Let $[(l, L), f] \in U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$. We wish to construct a degree-d map on the fibre of $p_n^* VU(n)$ over $[(l, L), f]$. To do this, recall that the orthonormal frame L defines coordinates on l via the assignment

$$(z_1, \ldots, z_n) \overset{L}{\longmapsto} z_1\, L(e_1) + \cdots + z_n\, L(e_n),$$

where $e_1, \ldots, e_n$ the standard basis of $\mathbf{C}^n$. These coordinates let us identify S^{2n-1} with its image $L(S^{2n-1}) \subseteq l$. Hence, we can realise f in the equivalence class $[(l, L), f]$ as a map $S^{2n-1} \to S(l)$, or more precisely, there is an induced map

$$\widetilde{f_l} = L \circ f : S^{2n-1} \longrightarrow S(l), \quad \text{where} \quad S(l) := L(S^{2n-1}) \subseteq l \tag{4.1}$$

fitting into the following diagram:

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "\widetilde{f_l}"] \arrow[d, equal] & S(l) \\
S^{2n-1} \arrow[r, "f"'] & S^{2n-1}. \arrow[u, "L"']
\end{tikzcd}$$

The induced $\widetilde{f_l}$ is well-defined on each equivalence class $[(l, L), f]$ due to the commutativity of the following diagram:

$$\begin{tikzcd}
S^{2n-1} \arrow[rr, "\widetilde{f_l}"] \arrow[dd, equal] \arrow[dr, equal] & & S(l) \arrow[dr, equal] & \\
& S^{2n-1} \arrow[rr, "(\widetilde{g^{-1}\circ f})_l = \widetilde{f_l}"] \arrow[dd, equal] & & S(l) \\
S^{2n-1} \arrow[rr, "f"] \arrow[dr, equal] & & S^{2n-1} \arrow[uu, "L"'] \arrow[dr, "g^{-1}"] & \\
& S^{2n-1} \arrow[rr, "g^{-1}\circ f"] & & S^{2n-1}. \arrow[uu, "L\circ g"']
\end{tikzcd}$$

Considering S^{2n-1} as the fibre of trivial bundle $S(\underline{\mathbf{C}}^n)$, the maps $\widetilde{f_l}$ fit together to define a fibrewise degree-d map

$$\begin{array}{rccc}
f_n^{\text{univ}} : & S(\underline{\mathbf{C}}^n) & \longrightarrow & S(p_n^* VU(n)) \\
& ([(l, L), f], z) & \longmapsto & ([(l, L), f], \widetilde{f_l}(z))
\end{array}$$

from the trivial sphere bundle to the sphere bundle of $p_n^* VU(n)$. That is to say, on each fibre over $[(l, L), f]$, f_n^{univ} restricts to $\widetilde{f_l}$ to give the following commutative diagram:

$$\begin{tikzcd}
S^{2n-1} \arrow[rr, "\widetilde{f_l}"] \arrow[d, hook] & & S(l) \arrow[d, hook] \\
S(\underline{\mathbf{C}}^n) \arrow[rr, "f_n^{\text{univ}}"] \arrow[dr] & & S(p_n^* VU(n)) \arrow[dl] \\
& U(n)\backslash\backslash \mathrm{Map}(S^{2n-1}, S^{2n-1})_d. &
\end{tikzcd}$$

4.2 The space $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ as a classifying space

In this section, we exhibit that $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ is the classifying space for the following Brown functor.

Definition 4.2.1. Let X be a connected compact Hausdorff space. We define

$$\mathcal{F}^{\mathrm{ts}}_{d,n}(X) = \{\, f^{\mathrm{ts}} : S(\underline{\mathbf{C}}^n) \to S(F^n) \,\}/\simeq$$

to be the set of homotopy equivalence classes of fibrewise degree-d maps between rank n vector bundles over X where the source is trivial. In particular, this defines a functor

$$\mathcal{F}^{\mathrm{ts}}_{d,n} : \mathsf{KHaus}^{\mathrm{op}} \longrightarrow \mathsf{Sets}.$$

The functor $\mathcal{F}^{\mathrm{ts}}_{d,n}$ (when restricted to CW-complexes) satisfies the conditions of Brown's Representability Theorem [Bro62], and therefore is representable by a *classifying space*. We ambiguously denote models for this classifying space by $(QS^0/U)^{\mathrm{ts}}_{d,n}$, which is well-defined up to homotopy type.

Similarly, we can define $\mathcal{F}^{\mathrm{tt}}_{d,n}$ where we trivialise the target instead.

Theorem 4.2.2 (A classifying space for $\mathcal{F}^{\mathrm{ts}}_{d,n}$). *The space $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ is a model for the classifying space $(QS^0/U)^{\mathrm{ts}}_{d,n}$ of $\mathcal{F}^{\mathrm{ts}}_{d,n}$, i.e., there is a natural bijection*

$$\begin{array}{ccc} [X, U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d] & \longrightarrow & \mathcal{F}^{\mathrm{ts}}_{d,n}(X) \\ [\Phi] & \longmapsto & \Phi^* f^{\mathrm{univ}}_n \end{array}$$

for connected compact Hausdorff spaces X.

Proof. We need to prove surjectivity and injectivity. We begin with surjectivity.

Surjectivity. Let $f : S(\underline{\mathbf{C}}^n) \to S(F)$ be a fibrewise degree-d map over X. Because $F \to X$ is a rank n vector bundle over X, there is a homotopy class $[\phi] \in [X, BU(n)]$ such that $F \cong \phi^* VU(n)$, i.e., ϕ is the classifying map of F:

$$\begin{tikzcd} F \arrow[r, "\cong"] \arrow[dr] & \phi^* VU(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & VU(n) \arrow[d] \\ & X \arrow[r, "\phi"'] & BU(n). \end{tikzcd}$$

WLOG, we identify F with the isomorphic bundle $\phi^* VU(n)$. We wish to create a lift Φ of ϕ to $U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ fitting into the following diagram:

$$\begin{tikzcd} & U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \arrow[d, "p_n"] \\ X \arrow[ur, dashed, "\Phi"] \arrow[r, "\phi"'] & BU(n). \end{tikzcd}$$

Define

$$\Phi : X \longrightarrow U(n)\backslash\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d, \quad \Phi(x) = [((\phi^* VU(n))_x, L), L^{-1} \circ f_x]. \tag{4.2}$$

By this, we mean:

- $l = (\phi^* VU(n))_x \subseteq \mathbf{C}^\infty$ is the n-dimensional subspace of $\mathbf{C}^\infty$ corresponding to the fibre of $\phi^* VU(n)$ over $x \in X$;
- $L : \mathbf{C}^n \to l$ is an orthonormal frame for l; and
- $f_x : S^{2n-1} \to S(l)$ is the restriction of f to the fibre over $x \in X$, a degree-d map of spheres, where we identify the fibre $S(\underline{\mathbf{C}}^n)_x$ of the trivial sphere bundle with S^{2n-1}.

Note that the definition of Φ does not depend on the choice of L, for if $L' : \mathbf{C}^n \to l$ is another orthonormal frame for l, then $g = L^{-1}L' \in U(n)$ so that

$$\begin{aligned}[((\phi^* VU(n))_x, L), L^{-1} \circ f_x] &= [((\phi^* VU(n))_x, L \circ g), g^{-1} \circ L^{-1} \circ f_x] \\ &= [((\phi^* VU(n))_x, L'), \circ L'^{-1} \circ f_x].\end{aligned}$$

By definition of the pullback for fibrewise degree-d maps, the pullback of $f_n^{\mathrm{univ}} : S(\underline{\mathbf{C}}^n) \to S(p_n^* VU(n))$ along Φ is a map

$$\Phi^* f_n^{\mathrm{univ}} : S(\Phi^* \underline{\mathbf{C}}^n) \longrightarrow S(\Phi^* p_n^* VU(n)).$$

Now, the pullback of the trivial bundle $\underline{\mathbf{C}}^n \to BU(n)$ is again a trivial bundle, this time over X. As for $\Phi^* p_n^* VU(n) \to X$, we have by functoriality that

$$\Phi^* p_n^* = (p_n \Phi)^* = \phi^*,$$

and therefore $\Phi^* p_n^* VU(n)$ is precisely $\phi^* VU(n)$. On each fibre, $\Phi^* f_n^{\mathrm{univ}}$ is defined to be the map

$$(\Phi^* f_n^{\mathrm{univ}})_x = (f_n^{\mathrm{univ}})_{\Phi(x)} : S^{2n-1} \longrightarrow S(\phi^* VU(n))_x.$$

But recall from (4.2) that $\Phi(x)$ is given by $[((\phi^* VU(n))_x, L), L^{-1} \circ f_x]$, where the $\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ coordinate is the composition $L^{-1} \circ f_x$. From our definition of f_n^{univ}, its restriction to the fibre over $[((\phi^* VU(n))_x, L), L^{-1} \circ f_x]$ is precisely $L \circ L^{-1} \circ f_x = f_x$ (see equation (4.1)). Hence, $(\Phi^* f_n^{\mathrm{univ}})_x = f_x$ for each $x \in X$, and therefore $\Phi^* f_n^{\mathrm{univ}} = f$.

This shows surjectivity of $[X, U(n)\backslash\!\backslash \mathrm{Map}(S^{2n-1}, S^{2n-1})_d] \to \mathcal{F}_{d,n}^{\mathrm{ts}}(X)$.

Injectivity. For injectivity, we verify homotopy of classifying maps in two steps:

1. First for homotopies of fibrewise degree-d maps (see Definition 2.3.5).
2. Then for isomorphisms of fibrewise degree-d maps (see Definition 2.3.4).

This will be sufficient, for the two operations above generate the equivalence relation of homotopy equivalence of fibrewise degree-d maps (see Definition 2.3.7).

Claim 4.2.3 (Homotopy of fibrewise degree-d maps). *Let $\Phi_0, \Phi_1 : X \to U(n)\backslash\!\backslash \mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ be two maps, and denote $f_i = \Phi_i^* f_n^{\mathrm{univ}}$, $i = 0, 1$, for their pullback fibrewise degree-d maps. Suppose there exists a homotopy between f_0 and f_1, i.e., a fibrewise degree-d map $f : S(\underline{\mathbf{C}}^n) \to S(F)$ over $X \times I$ such that f_i are the restrictions of f to $X \times \{i\}$, $i = 0, 1$. Then there exists a homotopy Φ_t between Φ_0 and Φ_1.*

Proof. We inspect the proof of surjectivity more carefully. Denote $\phi_i = p_n\Phi_i : X \to BU(n)$ for $i = 0, 1$: these are the classifying maps for the target bundles of $\Phi_i^* f_n^{\mathrm{univ}}$, $i = 0, 1$. We have commutative diagrams

$$\begin{tikzcd} & U(n)\backslash\!\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \arrow[d, "p_n"] \\ X \arrow[ur, "\Phi_i"] \arrow[r, "\phi_i"'] & BU(n), \end{tikzcd}$$

for $i = 0, 1$. Hence, we see that the lifts Φ_i must be of the form (4.2) constructed for the proof of surjectivity. But now, since the pullbacks $\phi_i^* VU(n) = F|_{X\times\{i\}}$, $i = 0, 1$, by assumption, $\phi_0^* VU(n)$ and $\phi_1^* VU(n)$ are isomorphic as vector bundles [Hat17, Proposition 1.7]. So there exists a homotopy ϕ_t from ϕ_0 to ϕ_1: in fact, this homotopy can be taken to be such that $\phi_t^* VU(n) = F|_{X\times\{t\}}$. We now need to lift this homotopy such that the lifts of the two ends ϕ_i coincide with the pre-existing lifts Φ_i, $i = 0, 1$:

$$\begin{tikzcd} & U(n)\backslash\!\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d \arrow[d, "p_n"] \\ X \arrow[ur, dashed, "\Phi_t"] \arrow[r, "\phi_t"'] & BU(n). \end{tikzcd}$$

But because we have the "interpolating" fibrewise degree-d map $f : S(\underline{\mathbf{C}}^n) \to S(F)$ over $X \times I$, this lift can be constructed exactly as in (4.2). We define

$$\Phi_t(x) = [((\phi_t^* VU(n))_x, L), L^{-1} \circ f_{x,t}].$$

By construction, Φ_t agrees with Φ_0 and Φ_1 at $t = 0, 1$. ∎

Dealing with isomorphisms will use the above claim.

Claim 4.2.4 (Isomorphism of fibrewise degree-d maps). *Let $\Phi_0, \Phi_1 : X \to U(n)\backslash\!\backslash\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ be two maps, and denote $f_i = \Phi_i^* f_n^{\mathrm{univ}}$, $i = 0, 1$, for their pullback fibrewise degree-d maps. Suppose there exists an isomorphism between f_0 and f_1. Then there exists a homotopy Φ_t between Φ_0 and Φ_1.*

Proof. Again, denote $\phi_i = p_n\Phi_i : X \to BU(n)$ for $i = 0, 1$. By definition, the data of a fibrewise degree-d map isomorphism is a commutative diagram

$$\begin{tikzcd} S(\underline{\mathbf{C}}^n) \arrow[rr, "f_0"] \arrow[dr, "g", "\cong"'] \arrow[ddr] & & S(\phi_0^* VU(n)) \arrow[dr, "h", "\cong"'] \arrow[ddl] & \\ & S(\underline{\mathbf{C}}^n) \arrow[rr, "f_1"] \arrow[ddr] & & S(\phi_1^* VU(n)) \arrow[ddl] \\ & X \arrow[dr, equal] & & \\ & & X. & \end{tikzcd}$$

for $U(n)$-bundle isomorphisms g and h. But with this data, we can construct a fibrewise degree-d map over $X \times I$ as follows:

1. Over $X \times [0, 1/2]$, we take $f_0 \times \mathrm{id} : S(\underline{\mathbf{C}}^n) \times [0, 1/2] \to S(\phi_0^* VU(n)) \times [0, 1/2]$.
2. Over $X \times [1/2, 1]$, we take $f_1 \times \mathrm{id} : S(\underline{\mathbf{C}}^n) \times [1/2, 1] \to S(\phi_0^* VU(n)) \times [1/2]$.
3. We glue $f_0 \times \mathrm{id}$ to $f_1 \times \mathrm{id}$ using the $U(n)$-bundle isomorphisms g on the source bundle and h on the target bundle.

Calling this new fibrewise degree-d map f, f satisfies the hypotheses of Claim 4.2.3 by construction. And hence we obtain the desired homotopy Φ_t. ■

The above two claims show injectivity of $[X, U(n)\backslash\backslash \mathrm{Map}(S^{2n-1}, S^{2n-1})_d] \to \mathcal{F}^{\mathrm{ts}}_{d,n}(X)$. □

Remark 4.2.5. We remark that the above proof applies to $\mathcal{F}^{\mathrm{tt}}_{d,n}$ where we have chosen to trivialise the target bundle. However, the proof does not depend actually on this choice. It is equally possible to choose to trivialise the source bundle, proving an analogous result for $\mathcal{F}^{\mathrm{ts}}_{d,n}$. In this case, the classifying space would be constructed by taking the homotopy orbit space $\mathrm{Map}(S^{2n-1}, S^{2n-1})_d /\!/ U(n)$ of $\mathrm{Map}(S^{2n-1}, S^{2n-1})_d$ under the *right* $U(n)$-action.

4.3 A stable viewpoint

In this section, we put Theorem 4.2.2 (A classifying space for $\mathcal{F}^{\mathrm{ts}}_{d,n}$) into the stable setting, and provide a different proof under this setting by looking at homotopy groups.

Definition 4.3.1. Let X be a connected compact Hausdorff space. We define

$$\mathcal{F}_d(X) = \{\, f : S(E) \to S(F)\,\}/\simeq_s$$

to be the set of stable homotopy equivalence classes of fibrewise degree-d maps between vector bundles over X, i.e., it is the stable equivalent of $\mathcal{F}^{\mathrm{ts}}_{d,n}(X)$. This defines a functor

$$\mathcal{F}_d : \mathsf{KHaus}^{\mathrm{op}} \longrightarrow \mathsf{Sets}.$$

The functor $\mathcal{F}_d$ (restricted to CW-complexes) satisfies the conditions of Brown's Representability Theorem [Bro62] (c.f. Definition 4.2.1), and therefore is representable by a *classifying space*. We denote models for the *classifying space* of $\mathcal{F}_d$ by $(QS^0/U)_d$, following [BM76, §4].

Remark 4.3.2 (The functor Q). Recall that given based space X, the functor Q applied to X is defined to be the direct limit

$$QX := \varinjlim_n \Omega^n \Sigma^n X,$$

where ΩX is loop space of X, and ΣX is the (reduced) suspension of X.

Applied to the two point space S^0, we have homotopy equivalences $\Sigma^n X \simeq S^n$, and $\Omega^n S^n \simeq \mathrm{Map}_*(S^n, S^n)$, and therefore

$$QS^0 \simeq \varinjlim_n \mathrm{Map}_*(S^n, S^n)$$

is the direct limit of the space of based maps $S^n \to S^n$. The subspace QS^0_d is the degree d component of QS^0, obtained as a direct limit of the degree d components $\mathrm{Map}_*(S^n, S^n)_d$ of $\mathrm{Map}_*(S^n, S^n)$. Equivalently, from the fibration sequence

$$\mathrm{Map}_*(S^n, S^n)_d \hookrightarrow \mathrm{Map}(S^n, S^n)_d \xrightarrow{\mathrm{ev}} S^n,$$

where ev is the evaluation map at a chosen basepoint of S^n, we find that the inclusion $\mathrm{Map}_*(S^n, S^n)_d \hookrightarrow \mathrm{Map}(S^n, S^n)_d$ induces isomorphisms on π_k for all $k < n$. Therefore, in the limit as $n \to \infty$, we have a homotopy equivalence

$$QS^0_d \simeq \varinjlim_n \mathrm{Map}(S^n, S^n)_d.$$

We will instead take this direct limit to be our definition of QS^0_d.

Now, by taking the limit

$$U\backslash\!\backslash QS^0_d = \varinjlim_n U(n)\backslash\!\backslash \mathrm{Map}(S^{2n-1}, S^{2n-1})_d,$$

our construction of the finite-rank universal fibrewise degree-d maps $f_n^{\mathrm{univ}} : S(\underline{\mathbf{C}}^n) \to S(p_n^* VU(n))$ defines a universal fibrewise degree-d map over $U\backslash\!\backslash QS^0_d$ which we will denote by $f^{\mathrm{univ}} : S(\underline{\mathbf{C}}^\infty) \to S(p^*VU)$, where $p = \lim_n p_n : U\backslash\!\backslash QS^0_d \to BU$:

$$\begin{tikzcd}
S^\infty \arrow[rr, "\tilde{f}_l"] \arrow[d, hook] & & S(l) \arrow[d, hook] \\
S(\underline{\mathbf{C}}^\infty) \arrow[rr, "f^{\mathrm{univ}}"] \arrow[dr] & & S(p^*VU) \arrow[dl] \\
& U\backslash\!\backslash QS^0_d. &
\end{tikzcd}$$

Theorem 4.3.3 (A classifying space for $\mathcal{F}_d$). *The space $U\backslash\!\backslash QS^0_d$ is a model for the classifying space, $(QS^0/U)_d$, and the classifying map $\phi_{f^{\mathrm{univ}}} : U\backslash\!\backslash QS^0_d \to (QS^0/U)_d$ for f^{univ} is a homotopy equivalence.*

Proof. We begin by remarking that bundle $p^*VU \to U\backslash\!\backslash QS^0_d$ is the pullback along the projection map $p : U\backslash\!\backslash QS^0_d \to BU$:

$$\begin{tikzcd}
p^*VU \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & VU \arrow[d] \\
U\backslash\!\backslash QS^0_d \arrow[r, "p"'] & BU.
\end{tikzcd}$$

There is a map $i : (QS^0/U)_d \to BU$ classifying for a fibrewise degree-d map $f : S(E) \to S(F)$ over X the bundle difference $[F - E] \in \widetilde{K}(X)$; that is, the pullback of VU along composition

$$X \xrightarrow{\phi_f} (QS^0/U)_d \xrightarrow{i} BU$$

gives the reduced K-theory element $[F-E]$, where ϕ_f is the classifying map for f. The $(QS^0/O)_d \to BU$ analogue is described by in [BM76, §4]. The map i gives rise to a fibration sequence

$$QS^0_d \xrightarrow{j} (QS^0/U)_d \xrightarrow{i} BU.$$

Here, we see that QS_d^0 is the fibre of i because the classifying maps $\phi_f : X \to (QS^0/U)_d$ which become null-homotopic after composing with i correspond precisely to the stable homotopy class of fibrewise degree-d maps $f : S(E) \to S(F)$ such that $[F - E] = 0 \in \widetilde{K}(X)$, i.e., f is stably isomorphic to $t : S(\underline{\mathbf{C}}^n) \to S(\underline{\mathbf{C}}^n)$ for large n. Via adjunction, t is equivalent to a map $X \to QS_d^0$.

First, we check the commutativity (up to homotopy) of the following square:

$$\begin{tikzcd}
U\backslash\backslash QS_d^0 \arrow[r, "\phi_{f^{\mathrm{univ}}}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft ?"] & (QS^0/U)_d \arrow[d, "i"] \\
BU \arrow[r, equal] & BU.
\end{tikzcd}$$

The composition $i\phi_{f^{\mathrm{univ}}} : U\backslash\backslash QS_d^0 \to BU$ going around the top right of the diagram corresponds to the K-theory class $[p^*VU - \underline{\mathbf{C}}^\infty] = [p^*VU] \in \widetilde{K}(U\backslash\backslash QS_d^0)$ by definition of i. But p^*VU is the pullback of the universal bundle $VU \to BU$ along $p : U\backslash\backslash QS_d^0 \to BU$, and p is tautologically classifying map of p^*VU. Being the vertical map on the left of the diagram, it yields the same K-theory class $[p^*VU] \in \widetilde{K}(U\backslash\backslash QS_d^0)$. So up to homotopy, the diagram commutes.

We now need to check that square for the induced map on fibres commutes (up to homotopy):

$$\begin{tikzcd}
QS_d^0 \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowleft ?"] & QS_d^0 \arrow[d, "j"] \\
U\backslash\backslash QS_d^0 \arrow[r, "\phi_{f^{\mathrm{univ}}}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & (QS^0/U)_d \arrow[d, "i"] \\
BU \arrow[r, equal] & BU
\end{tikzcd}$$

But this follows from the same argument as above, where now we argue instead for a map $g : X \to U\backslash\backslash QS_d^0$, which is null-homotopic after composing with p. If $pg \simeq \text{const} : X \to BU$, then $[pg] \in [X, BU]$ corresponds to the element $0 \in \widetilde{K}(X)$, and therefore the pullback $g^* f^{\mathrm{univ}}$ belongs to the the stable homotopy class of fibrewise degree-d maps $S(\underline{\mathbf{C}}^n) \to S(\underline{\mathbf{C}}^n)$ for large n. In other words, g factors through the fibre QS_d^0 of $U\backslash\backslash QS_d^0$. But now, by homotopy commutativity of the bottom square, the composition $i\phi_{f^{\mathrm{univ}}} g \simeq \text{const} : X \to BU$ is also null-homotopic, corresponding to the same element $0 \in \widetilde{K}(X)$, and so $\phi_{f^{\mathrm{univ}}} g$ belongs the same stable homotopy class of fibrewise degree-d maps $S(\underline{\mathbf{C}}^n) \to S(\underline{\mathbf{C}}^n)$ for large n: it also factors through the fibre QS_d^0 of $(QS^0/U)_d$:

$$\begin{tikzcd}
 & QS_d^0 \arrow[r, equal] \arrow[d, dashed] & QS_d^0 \arrow[d] \\
X \arrow[ur, dashed, bend left, "g"] \arrow[urr, dashed, "\phi_{f^{\mathrm{univ}}} g"] \arrow[r, "g"] \arrow[dr, "\text{const}"'] & U\backslash\backslash QS_d^0 \arrow[r, "\phi_{f^{\mathrm{univ}}}"] \arrow[d, "p"] & (QS^0/U)_d \arrow[d, "i"] \\
 & BU \arrow[r, equal] & BU.
\end{tikzcd}$$

Hence, we also have commutativity of the map on the fibres.

Now applying π_k to the above diagram, we have by the long exact sequence for a fibration

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\pi_{k+1}(BU) \arrow[r, equal] \arrow[d] & \pi_{k+1}(BU) \arrow[d] \\
\pi_k(QS^0_d) \arrow[r, equal] \arrow[d] & \pi_k(QS^0_d) \arrow[d, "j_*"] \\
\pi_k(U\backslash\!\backslash QS^0_d) \arrow[r, "\phi_{f^{\mathrm{univ}}*}"] \arrow[d, "p_*"'] & \pi_k((QS^0/U)_d) \arrow[d, "i_*"] \\
\pi_k(BU) \arrow[r, equal] \arrow[d] & \pi_k(BU) \arrow[d] \\
\pi_{k-1}(QS^0_d) \arrow[r, equal] \arrow[d] & \pi_{k-1}(QS^0_d) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

for all k. Since 4 out of 5 of the above horizontal maps are isomorphisms, by the 5-lemma, $\phi_{f^{\mathrm{univ}}} : U\backslash\!\backslash QS^0_d \to (QS^0/U)_d$ induces an isomorphism on all π_k, i.e., it is a weak homotopy equivalence. The classifying space given by Brown's Representability Theorem [Bro62] has the homotopy type of a CW-complex, and the homotopy orbit space $U\backslash\!\backslash QS^0_d$ is also a CW-complex. So by Whitehead's Theorem [Hat01, Theorem 4.5], it is a homotopy equivalence. $\square$

Remark 4.3.4. As for the finite-dimensional case, it is possible to prove analogously that the homotopy orbit space $QS^0_d/\!\!/U$ obtained through the limit

$$QS^0_d/\!\!/U = \varinjlim_n \mathrm{Map}(S^{2n-1}, S^{2n-1})_d/\!\!/U(n)$$

is also a model for the classifying space of $\mathcal{F}_d$. Hence, in fact, the two homotopy orbit spaces $U\backslash\!\backslash QS^0_d$ and $QS^0_d/\!\!/U$ by the left and right actions on QS^0_d respectively are homotopy equivalent.

5 The $\mathcal{A}$-space as a classifying space

Recall that Theorem 2.2.3 provides us with a map $N : \mathrm{Map}_{0,\infty}(\widehat{\mathbf{C}}^{n+1}, \widehat{\mathbf{C}}^{n+1}) \to \mathrm{Map}(S^{2n+1}, S^{2n+1})$. By Remark 2.2.4, the degree-d split polynomial space $SP(n)_d$ has a map induced by N into the space $\mathrm{Map}(S^{2n+1}, S^{2n+1})_d$. Now, the $U(n+1)$-action on $SP(n)_d$ also allows us to define a homotopy orbit space as follows (c.f. Definition 4.1.1).

Definition 5.0.1. The *homotopy orbit space* $U(n+1)\backslash\!\!\backslash SP(n)_d$ is defined to be the balanced product

$$U(n+1)\backslash\!\!\backslash SP(n)_d := EU(n+1) \underset{U(n+1)}{\times} SP(n)_d$$

under the usual right $U(n+1)$-action on $EU(n+1)$ and the *free* left $U(n+1)$-action on $SP(n)_d$.

The map $SP(n)_d \to \mathrm{Map}(S^{2n+1}, S^{2n+1})_d$ induces a map on the homotopy orbits $U(n+1)\backslash\!\!\backslash SP(n)_d \to U(n+1)\backslash\!\!\backslash \mathrm{Map}(S^{2n+1}, S^{2n+1})_d$. Taking the limit $n \to \infty$, we obtain maps

$$U\backslash\!\!\backslash SP_d = \varinjlim_n U(n+1)\backslash\!\!\backslash SP(n)_d \longrightarrow U\backslash\!\!\backslash QS^0_d = \varinjlim_n U(n+1)\backslash\!\!\backslash \mathrm{Map}(S^{2n+1}, S^{2n+1})_d.$$

In light of our construction in Chapter 4, we give a vague conjecture about $U\backslash\!\!\backslash SP_d$.

Conjecture 5.0.2. *$U\backslash\!\!\backslash SP_d$ is the classifying space for fibrewise degree-d split polynomial maps.*

We leave investigating this result for future study.

To further motivate the study of the space $U\backslash\!\!\backslash SP_d$, we introduce some theory from the work of Crowley and Nagy [CN23]. Given a line bundle $L \to Y$ over a smooth manifold Y, a divisor D_L of L is a transverse intersection of the zero section of L with itself. Now let $\gamma_n \to \mathbf{C}P^n$ denote the tautological line bundle over $\mathbf{C}P^n$. For a multidegree $\underline{d} = \{d_1, \ldots, d_k\}$, the complete intersection $X_n(\underline{d})$ is a divisor of the bundle $\gamma_n^{\oplus d_1} \oplus \cdots \oplus \gamma_n^{\oplus d_k}$. Using the theory of branched coverings, one can construct a canonical *normal map*

$$\hat{\eta}(\underline{d}) = (\gamma_n \to \gamma_n^{\oplus d_1}) \oplus \cdots \oplus (\gamma_n \to \gamma_n^{d_k}),$$

which is the direct sum of fibrewise degree-d_i maps. The total degree is the product $d := d_1 \cdots d_k$. By our Theorem 4.3.3 (A classifying space for $\mathcal{F}_d$), $\hat{\eta}(\underline{d})$ defines a homotopy class of maps $[\eta(\underline{d}) : \mathbf{C}P^n \to U\backslash\!\!\backslash QS^0_d]$, called the *normal invariant* of $X_n(\underline{d})$. However, we notice that by construction, $\hat{\eta}(\underline{d})$ is a fibrewise split polynomial, and hence the normal invariant a posteriori is a homotopy class of maps $[\eta(\underline{d}) : \mathbf{C}P^n \to U/\!\!/SP_d]$:

$$\begin{tikzcd}
\mathbf{C}P^n \arrow[rr, "\eta(\underline{d})"] \arrow[dr, dashed, "\eta(\underline{d})"'] & & U\backslash\!\!\backslash QS^0_d \\
& U\backslash\!\!\backslash SP_d. \arrow[ur, hook] &
\end{tikzcd}$$

It is conjectured by Crowley and Nagy that if two normal invariants are homotopic in $U\backslash\!\!\backslash QS^0_d$, then they are already homotopic in $U\backslash\!\!\backslash SP_d$.

In this chapter, we study some properties of $U\backslash\!\!\backslash SP_d$ and the related $\mathcal{A}$-space.

5.1 The $\mathcal{A}$-space and the homotopy quotient $U(n+1)\backslash\backslash SP(n)_d$

We actually have the following relationship between $U(n+1)\backslash\backslash SP(n)_d$ and the related $\mathcal{A}$-space $\mathcal{A}(n)_d$, a consequent of $SP(n)_d$ having a free $U(n+1)$ action.

Theorem 5.1.1. *The homotopy orbit space $U(n+1)\backslash\backslash SP(n)_d$ is homotopy equivalent to the degree-d $\mathcal{A}$-space $\mathcal{A}(n)_d$.*

Proof. From the Borel construction with the left $U(n+1)$-action on $SP(n)_d$, we can construct two different fibration sequences:

$$\begin{tikzcd} & SP(n)_d \arrow[d, hook] & \\ EU(n+1) \arrow[r, hook] & U(n+1)\backslash\backslash SP(n)_d \arrow[r] \arrow[d] & \mathcal{A}(n)_d \\ & BU(n+1). & \end{tikzcd}$$

The vertical sequence is the canonical fibration sequence of a homotopy orbit space. The horizontal sequence arises from the fact that the left $U(n+1)$-action is *free* (recall Section 3.2.3): There is always a projection $U(n+1)\backslash\backslash SP(n)_d \to \mathcal{A}(n)_d$ onto the quotient $\mathcal{A}(n)_d = U(n+1)\backslash SP(n)_d$, but the fibre is $EU(n+1)$ when the action is free. To see this, fix some basepoint $[f] \in \mathcal{A}(n)_d$. Now note that the preimages of $[f]$ in $U(n+1)\backslash\backslash SP(n)_d$ are orbits $[e', f']$ of points $(e', f') \in EU(n+1) \times SP(n)_d$ under the $U(n+1)$-action such that $[f'] = [f]$. By freeness of $U(n+1)$ acting on $SP(n)_d$, each orbit $[e, f']$ has a canonical representative $[e, f]$ where the $SP(n)_d$ coordinate is exactly f. This identifies the fibre with $EU(n+1)$.

Now, because the fibre $EU(n+1)$ is a contractible space, we have by the homotopy long exact sequence for a fibration

$$\cdots \longrightarrow \pi_k(EU(n+1)) \longrightarrow \pi_k(U(n+1)\backslash\backslash SP(n)_d) \longrightarrow \pi_k(\mathcal{A}(n)_d) \longrightarrow \pi_{k-1}(EU(n+1)) \longrightarrow \cdots,$$

where $\pi_k(EU(n+1)) = 0$ for all k. Hence, the projection $U(n+1)\backslash\backslash SP(n)_d \to \mathcal{A}(n)_d$ induces isomorphisms on all homotopy groups, i.e., it is a weak homotopy equivalence. Since all spaces in question are CW-complexes, by Whitehead's Theorem [Hat01, Theorem 4.5], this induces a homotopy equivalence $U(n+1)\backslash\backslash SP(n)_d \simeq \mathcal{A}(n)_d$. □

Corollary 5.1.1. *The homotopy orbit space $U\backslash\backslash SP_d$ is homotopy equivalent to the stable degree-d $\mathcal{A}$-space $\mathcal{A}_d$.*

Proof. This follows from Theorem 5.1.1 after taking the limit $n \to \infty$. □

5.2 Vector bundles over the $\mathcal{A}$-space

Let $\rho : SP(n)_d \to \mathcal{A}(n)_d$ denote the quotient map of $SP(n)_d$ by the left $U(n+1)$-action. Recall that the left action is free, and so ρ is a principal $U(n+1)$-bundle. It has an associated vector bundle, which we denote by $V\rho : V(SP(n)_d) \to \mathcal{A}(n)_d$. We are interested in the isomorphism class of $V\rho$. In this section, we will study restrictions of $V\rho$ to certain subspaces of $\mathcal{A}(n)_d$.

5.2.1 The bundle $V\rho$ over the maximal anti-diagonal

For the rest of this section, we assume that $k \leqslant n + 1$ is a positive integer.

We define the *anti-diagonal* of the product $(\mathbf{C}P^n)^{\times k}$ to be the subspace

$$\Delta^-_{\mathbf{C}P^n} = \{ ([v_1], \ldots, [v_k]) \in (\mathbf{C}P^n)^{\times k} \mid v_i \perp v_j \text{ for all } i \neq j \}.$$

There are k orthogonal line bundles over $\Delta^-_{\mathbf{C}P^n}$. These are the tautological line bundles whose total spaces, as subspaces of $\Delta^-_{\mathbf{C}P^n} \times \mathbf{C}^{n+1}$, are given by

$$\kappa_{i,n} = \{ (([v_1], \ldots, [v_k]), v) \mid ([v_1], \ldots, [v_k]) \in \Delta^-_{\mathbf{C}P^n},\ v \in \mathbf{C}v_i \},$$

where $i = 1, \ldots, k$. Each $\kappa_{i,n}$ comes equipped with a projection $\kappa_{i,n} \to \Delta^-_{\mathbf{C}P^n}$, which is the restriction of the projection $\Delta^-_{\mathbf{C}P^n} \times \mathbf{C}^{n+1} \to \Delta^-_{\mathbf{C}P^n}$. The direct sum of all of these line bundles has an orthogonal complement which also admits an explicit description: it is the bundle whose total space, as a subspace of $\Delta^-_{\mathbf{C}P^n} \times \mathbf{C}^{n+1}$, is given by

$$\begin{aligned}&(\kappa_{1,n} \oplus \cdots \oplus \kappa_{k,n})^\perp \\ &\quad = \{ (([v_1], \ldots, [v_k]), v) \mid ([v_1], \ldots, [v_k]) \in \Delta^-_{\mathbf{C}P^n},\ v \in (\mathbf{C}v_1 \oplus \cdots \oplus \mathbf{C}v_k)^\perp \}.\end{aligned}$$

Of course, we have the vector bundle isomorphism

$$\kappa_{1,n} \oplus \cdots \oplus \kappa_{k,n} \oplus (\kappa_{1,n} \oplus \cdots \oplus \kappa_{k,n})^\perp \cong \underline{\mathbf{C}}^{n+1},$$

where $\underline{\mathbf{C}}^{n+1}$ denotes the trivial rank $n + 1$ bundle over $\Delta^-_{\mathbf{C}P^n}$.

We now turn our attention to the $\mathcal{A}$-space.

Definition 5.2.1 (Maximal anti-diagonal of the $\mathcal{A}$-space). Let $d = p_1 \cdots p_k$ be a product of distinct primes. We define the *maximal anti-diagonal* of $\mathcal{A}(n)_d$ to be the subspace

$$\Delta^- = \{ [(v_1, p_1) \circ \cdots \circ (v_k, p_k)] \mid v_1, \ldots, v_k \in S^{2n+1},\ v_i \perp v_j \text{ for all } i \neq j \}.$$

By Proposition 3.2.2 (Commutativity of atomic split polynomials), (v_i, p_i) and (v_j, p_j) commute when $v_i \perp v_j$, and so their ordering in the composition does not matter: the vectors v_i are distinguished by the prime degree p_i of their atomic split polynomial map only. Therefore, there is a well-defined map $\Delta^- \to \Delta^-_{\mathbf{C}P^n}$ given by fixing a particular ordering of the prime factors:

$$[(v_1, p_1) \circ \cdots \circ (v_k, p_k)] \longmapsto ([v_1], \ldots, [v_k]).$$

This map is a homeomorphism by Proposition 3.4.1 (Relations in $\mathcal{A}(n)_d$). Hence, the tautological line bundles $\kappa_{i,n}$, $i = 1, \ldots, k$, can also be considered as line bundles over Δ^-. Alternatively, we give the explicit definition: the total spaces, as subspaces of $\Delta^- \times \mathbf{C}^{n+1}$, are given by

$$\kappa_{p_i,n} = \{ ([(v_1, p_1) \circ \cdots \circ (v_k, p_k)], v) \mid [(v_1, p_1) \circ \cdots \circ (v_k, p_k)] \in \Delta^-,\ v \in \mathbf{C}v_i \},$$

where $i = 1, \dots, k$. Note that here, we index by the prime p_i instead, as there is no particular ordering to the atomic split polynomials in the composition. Indeed, the orthogonal complement of the sum $\kappa_{p_1,n} \oplus \cdots \oplus \kappa_{p_k,n}$ is the bundle with total space

$$\begin{aligned}&(\kappa_{p_1,n} \oplus \cdots \oplus \kappa_{p_k,n})^\perp \\ &\quad = \{\, ([(v_1, p_1) \circ \cdots \circ (v_k, p_k)], v) \mid [(v_1, p_1) \circ \cdots \circ (v_k, p_k)] \in \Delta^-,\ v \in (\mathbf{C}v_1 \oplus \cdots \oplus \mathbf{C}v_k)^\perp \,\}.\end{aligned}$$

Remark 5.2.2. We call Δ^- the *maximal* anti-diagonal because there are subspaces of $\mathcal{A}(n)_d$ consisting of normal form factorisations where only *some* of the atomic maps are in orthogonal directions. These maps still commute, but only past each other, so that we can have situations where $v_1 \perp v_2$, $v_2 \perp v_3$, but $v_3 \not\perp v_1$, and therefore

$$v_1 \circ v_2 \circ v_3 = v_2 \circ v_1 \circ v_3 \neq v_2 \circ v_3 \circ v_1.$$

In Δ^-, all atomic maps commute, so the commutativity of the maps is “maximal” in this sense.

Theorem 5.2.3. *Let Δ^- be the maximal anti-diagonal of $\mathcal{A}(n)_d$, where $d = p_1 \cdots p_k$ is a product of distinct primes. Then there is a vector bundle isomorphism*

$$V\rho|_{\Delta^-} \cong \kappa_{p_1,n}^{\otimes p_1} \oplus \cdots \oplus \kappa_{p_k,n}^{\otimes p_k} \oplus (\kappa_{p_1,n} \oplus \cdots \oplus \kappa_{p_k,n})^\perp.$$

Proof. We consider the restriction of the principal $U(n+1)$-bundle $\rho : SP(n) \to \mathcal{A}(n)$ to the anti-diagonal Δ^-. The total space over Δ^- is the preimage

$$SP(n)_d^- = \{\, A \circ (v_1, p_1) \circ \cdots \circ (v_k, p_k) \mid A \in U(n+1),\ v_1, \dots, v_k \in S^{2n+1},\ v_i \perp v_j \text{ for all } i \neq j \,\}.$$

We begin by constructing a model for $SP(n)_d^-$ to gain a better understanding of its structure. Referring back to Proposition 3.2.2 (Commutativity of atomic split polynomials), commutativity of the atomic split polynomials in the normal form factorisation of $f \in SP(n)_d^-$ allows us to write

$$f = A \circ (v_1, p_1) \circ \cdots \circ (v_k, p_k)$$

for $A \in U(n+1)$, where we additionally impose the condition that the ordering of the prime degrees in the composition must be $(p_1, \dots, p_k)$ as written above. Under this condition, the normal form representation is unique up to the following relation (c.f. relation 6 in Relations in $SP(n)$):

$$A \circ (v_1, p_1) \circ \cdots \circ (v_k, p_k) = A A_{v_1}^{\lambda_k^{p_1 - 1}} \cdots A_{v_k}^{\lambda_1^{p_k - 1}} \circ (\lambda_1 v_1, p_1) \circ \cdots \circ (\lambda_k v_k, p_k)$$

for all $(\lambda_1, \dots, \lambda_k) \in T^k = (S^1)^{\times k}$. Let $\Delta_L^-(d_1, \dots, d_k)$ be the subspace of the product $L_{d_1}^{2n+1} \times \cdots \times L_{d_k}^{2n+1}$ consisting of elements $([v_1], \dots, [v_k])$ satisfying $v_i \perp v_j$ for all $i \neq j$, which we refer to as the *anti-diagonal* of $L_{d_1}^{2n+1} \times \cdots \times L_{d_k}^{2n+1}$. Write $\Delta_L^- = \Delta_L^-(p_1 - 1, \dots, p_k - 1)$. Then $SP(n)_d^-$ can be modelled as the twisted balanced product

$$SP(n)_d^- \cong U(n+1) \underset{T^k}{\times} \Delta_L^- := \frac{U(n+1) \times \Delta_L^-}{T^k},$$

where the T^k-action on Δ_L^- is given by

$$(\lambda_1, \dots, \lambda_k) \cdot (A, v_1, \dots, v_k) = \big(A A_{v_1}^{\lambda_k^{p_1 - 1}} \cdots A_{v_k}^{\lambda_1^{p_k - 1}}, \lambda_1 v_1, \dots, \lambda_k v_k\big).$$

By construction, $SP(n)^-_d$ has a left $U(n+1)$-action restricted from $SP(n)$, being the preimage of Δ^-. However, the right $U(n+1)$-action also stabilises $SP(n)^-_d$, for we have the equality

$$(v_1, p_1) \circ \cdots \circ (v_k, p_k) \circ A = A \circ (A^* v_1, p_1) \circ \cdots \circ (A^* v_k, p_k)$$

for all $A \in U(n+1)$ by relation 5 of Relations in $SP(n)$, it remains that $A^* v_i \perp A^* v_j$ for all $i \neq j$ because A is unitary. We now describe these actions on the homeomorphic space $U(n+1) \times_{T^k} \Delta^-_L$:

1. The *left* action, corresponding to pre-composition, is given by

$$\begin{array}{ccc} U(n+1) \times (U(n+1) \times_{T^k} \Delta^-_L) & \longrightarrow & U(n+1) \times_{T^k} \Delta^-_L \\ (g, \quad [A, v_1, \ldots, v_k]) & \longmapsto & [gA, v_1, \ldots, v_k]. \end{array}$$

   The quotient of $U(n+1) \times_{T^k} \Delta^-_L$ by this action is $\Delta^- \cong \Delta^-_{\mathbf{C}P^n}$, and the quotient map is

$$\begin{array}{rccc} \rho : & U(n+1) \times_{T^k} \Delta^-_L & \longrightarrow & \Delta^-_{\mathbf{C}P^n} \\ & [A, v_1, \ldots, v_k] & \longmapsto & ([v_1], \ldots [v_k]). \end{array}$$

   Concretely, ρ is the assignment of a split polynomial $f \in SP(n)^-_d$ to the collection of hyperplanes in its set of critical points in the domain, the irreducible components of $Z[f] \subseteq \mathbf{C}^{n+1}$ (c.f. equation (3.1)).

2. The *right* action, corresponding to post-composition, is given by

$$\begin{array}{ccc} (U(n+1) \times_{T^k} \Delta^-_L) \times U(n+1) & \longrightarrow & U(n+1) \times_{T^k} \Delta^-_L \\ ([A, v_1, \ldots, v_k], \quad g) & \longmapsto & [gA, g^{-1} v_1, \ldots, g^{-1} v_k]. \end{array}$$

   The quotient of $U(n+1) \times_{T^k} \Delta^-_L$ by this action is also $\Delta^- \cong \Delta^-_{\mathbf{C}P^n}$, and the quotient map is

$$\begin{array}{rccc} \sigma : & U(n+1) \times_{T^k} \Delta^-_L & \longrightarrow & \Delta^-_{\mathbf{C}P^n} \\ & [A, v_1, \ldots, v_k] & \longmapsto & ([Av_1], \ldots [Av_k]). \end{array}$$

   Concretely, σ is the assignment of a split polynomial $f \in SP(n)^-_d$ to the collection of hyperplanes in its set of critical values in the codomain.

We will see that the fibre of the right action defines a restriction of the $U(n+1)$-bundle under the left action.

Let $e_1, \ldots, e_{n+1}$ be the standard basis of $\mathbf{C}^{n+1}$, and let $F = \sigma^{-1}([e_1], \ldots, [e_k])$ denote the fibre of σ. Explicitly,

$$F = \{\, [A, \mu_1 A^* e_1, \ldots, \mu_k A^* e_k] \mid A \in U(n+1),\ \mu_1, \ldots, \mu_k \in S^1 \,\}.$$

This corresponds to the collection of split polynomials in $SP(n)^-_d$ whose set of critical values is the union of the lines $\mathbf{C}e_i \subseteq \mathbf{C}^{n+1}$, $i = 1, \ldots, k$. There is the following commutation relation:

$$A A^{\lambda}_{\mu A^* e_i} = A^{\lambda}_{e_i} A,$$

and $A^{\lambda_1}_{v_1}$ and $A^{\lambda_2}_{v_2}$ commute if $v_1 \perp v_2$. So, in fact,

$$[A, \mu_1 A^* e_1, \ldots, \mu_k A^* e_k] = \left[A^{\mu^{1-p_1}}_{e_1} \ldots A^{\mu^{1-p_k}}_{e_k} A, A^* e_1, \ldots, A^* e_k\right].$$

We now remark upon the following important fact: expressed in matrix form, A^*e_i is precisely the conjugate of the ith row of A. Therefore, writing $v_i = A^*e_i \in \mathbf{C}^{n+1}$ for $i = 1, \dots, n+1$, each element $[A, \mu_1 A^*e_1, \dots, \mu_k A^*e_k] \in F$ may be expressed as

$$[A, \mu_1 A^*e_1, \dots, \mu_k A^*e_k] = \left[\begin{pmatrix} \overline{v}_1^t \\ \vdots \\ \overline{v}_k^t \\ \overline{v}_{k+1}^t \\ \vdots \\ \overline{v}_{n+1}^t \end{pmatrix}, \mu_1 v_1, \dots, \mu_k v_k\right]$$

$$= \left[\begin{pmatrix} \mu^{1-p_1}\overline{v}_1^t \\ \vdots \\ \mu^{1-p_k}\overline{v}_k^t \\ \overline{v}_{k+1}^t \\ \vdots \\ \overline{v}_{n+1}^t \end{pmatrix}, v_1, \dots, v_k\right] = \left[A_{e_1}^{\mu^{1-p_1}} \dots A_{e_k}^{\mu^{1-p_k}} A, A^*e_1, \dots, A^*e_k\right],$$

where we have identified elements of $\mathbf{C}^{n+1}$ with column vectors and the symbol

$$\begin{pmatrix} \overline{v}_1^t \\ \vdots \\ \overline{v}_{n+1}^t \end{pmatrix}$$

denotes the $(n+1) \times (n+1)$ matrix whose rows are the transposes of the vectors $\overline{v}_1, \dots, \overline{v}_{n+1}$.

To understand the Δ_L^- component of the fibre F, let us consider it in isolation. Fix a choice of orthonormal vectors $v_{k+1}, \dots, v_{n+1} \in S^{2n+1}$, and for brevity we denote by $F(\overline{w}_1, \dots, \overline{w}_k, \mu_1, \dots, \mu_k)$ the element

$$\left[\begin{pmatrix} \overline{w}_1^t \\ \vdots \\ \overline{w}_k^t \\ \overline{v}_{k+1}^t \\ \vdots \\ \overline{v}_{n+1}^t \end{pmatrix}, \mu_1 w_1, \dots, \mu_k w_k\right] \in F,$$

where together, $w_1, \dots, w_k, v_{k+1}, \dots, v_{n+1}$ form an orthonormal basis of $\mathbf{C}^{n+1}$. We have the relation:

$$F(\overline{w}_1, \dots, \overline{w}_k, \mu_1, \dots, \mu_k) = F(\overline{\mu_1^{p_1-1} w_1}, \dots, \overline{\mu_k^{p_k-1} w_k}, 1, \dots, 1)$$

for all $(\mu_1, \dots, \mu_k) \in T^k$. Consequently, we can construct a mapping defined in the following way. Let the symbol $L_p(\overline{w}, \mu)$ denote the set

$$L_p(\overline{w}, \mu) = \{\, \nu\mu w \mid \nu^{1-p} w = \nu\mu w,\ \nu \in S^1 \,\} = \{\, \nu\mu w \mid \nu^p = \mu^{-1},\ \nu \in S^1 \,\}.$$

This set defines element of L_p^{2n+1}. We now have a mapping into the anti-diagonal $\Delta_L^-(p_1,\dots,p_k)$:

$$F(\overline{w}_1,\dots,\overline{w}_k,\mu_1,\dots,\mu_k)\longmapsto (L_{p_1}(\overline{w}_1,\mu_1),\dots,L_{p_k}(\overline{w}_k,\mu_k))\in\Delta_L^-(p_1,\dots,p_k). \tag{5.1}$$

This mapping is injective: to see this, suppose that

$$\{\,\nu\mu w\mid \nu^{1-p}w=\nu\mu w,\ \nu\in S^1\,\}=\{\,\nu'\mu'w'\mid \nu'^{1-p}w'=\nu'\mu'w',\ \nu'\in S^1\,\},$$

which means that there exist $\nu,\nu'\in S^1$ such that

$$\nu^{1-p}w=\nu\mu w=\nu'^{1-p}w'=\nu'\mu'w'.$$

And so

$$L_p(\overline{w},\mu)=L_p(\overline{\nu^{1-p}w},\nu\mu)=L_p(\overline{\nu'^{1-p}w'},\nu'\mu')=L_p(\overline{w}',\mu').$$

Applying this to all $L_{p_i}, i=1,\dots,k$, at once, we obtain injectivity of (5.1). Now letting $v_{k+1},\dots,v_{n+1}\in S^{2n+1}$ to vary, we find that there is a homeomorphism induced by (5.1) of F onto the subspace of the product

$$L_{p_1}^{2n+1}\times\cdots\times L_{p_k}^{2n+1}\times\underbrace{S^{2n+1}\times\cdots\times S^{2n+1}}_{n+1-k\text{ times}}$$

where the elements are represented by $n+1$ vectors $w_1,\dots,w_k,v_{k+1},\dots,v_{n+1}$ that form an orthonormal basis of $\mathbf{C}^{n+1}$. This subspace is a subbundle of the frame bundle of the claimed vector bundle $\kappa_{p_1,n}^{\otimes p_1}\oplus\cdots\oplus\kappa_{p_k,n}^{\otimes p_k}\oplus(\kappa_{p_1,n}\oplus\cdots\oplus\kappa_{p_k,n})^{\perp}$, where the first k vectors of the frame $w_i, i=1,\dots,k$, each lie in their respective subbundles $\kappa_{p_i,n}^{\otimes p_i}\subseteq\kappa$, $i=1,\dots,k$, (see Lemma 2.4.1) and the last $n+1-k$ vectors $v_i, i=k+1,\dots,n+1$, lie in the orthogonal complement $(\kappa_{p_1,n}\oplus\cdots\oplus\kappa_{p_k,n})^{\perp}$. We will use the notation $[w_1,\dots,w_k,v_{k+1},\dots,v_{n+1}]$ to denote an element of the fibre F.

Remark 5.2.4. If the $L_{p_i}^{2n+1}$ factors were instead spheres S^{2n+1}, then the subspace of $n+1$ vectors forming an orthonormal basis of $\mathbf{C}^{n+1}$ is the frame bundle of the trivial bundle

$$F(\underline{\mathbf{C}}^{n+1})\cong F(\kappa_{p_1,n}\oplus\cdots\oplus\kappa_{p_k,n}\oplus(\kappa_{p_1,n}\oplus\cdots\oplus\kappa_{p_k,n})^{\perp})$$

over the anti-diagonal. The lens space factors correspond to the twisting introduced by the tensor powers.

Now, F no longer has a left $U(n+1)$-action as it is not a stable subspace of $SP(n)_d^-$. But the stabiliser of F under the left $U(n+1)$-action is in fact the subgroup $U(1)^{\times k}\times U(n+1-k)\subseteq U(n+1)$ corresponding to those matrices which fix each line $[e_i]\in\mathbf{C}P^n$, $i=1,\dots,k$. To see this, let $g\in U(n+1)$ such that $g\cdot[A,v_1,\dots,v_k]\in F$ for all $[A,v_1,\dots,v_k]\in F$. This is precisely the condition that

$$\sigma([gA,v_1,\dots,v_k])=([gAv_1],\dots,[gAv_k])=([ge_1],\dots,[ge_k])\overset{!}{=}([e_1],\dots,[e_k]).$$

So the left $U(n+1)$-action on $SP(n)_d^-$ restricts to a left $U(1)^{\times k}\times U(n+1-k)$-action on F, whence we get a commutative diagram of pointed sets

$$\begin{tikzcd}
U(1)^{\times k} \times U(n+1-k) \arrow[r, hook] \arrow[d, hook] & U(n+1) \arrow[r, two heads] \arrow[d, hook] & \Delta^-_{\mathbf{C}P^n} \arrow[d, equal] \\
F \arrow[r, hook] \arrow[d, two heads, "\rho"'] & SP(n)^-_d \arrow[r, two heads, "\sigma"] \arrow[d, two heads, "\rho"'] & \Delta^-_{\mathbf{C}P^n} \arrow[d, two heads] \\
\Delta^- \arrow[r, equal] & \Delta^- \arrow[r, two heads] & *.
\end{tikzcd}$$

The vertical bundle $F \to \Delta^-$ on the left is a reduction of the structure group $U(n+1)$ of the middle vertical bundle $SP(n)^-_d \to \Delta^-$ to $U(1)^{\times k} \times U(n+1-k)$. Indeed, by Lemma 2.1.8 (Induced bundle), we have $U(n+1)$-bundle the isomorphism

$$F \underset{U(1)^{\times k} \times U(n+1-k)}{\times} U(n+1) \cong SP(n)^-_d.$$

On the level of associated vector bundles, the reduction means that $V\rho|_{\Delta^-}$ decomposes into the direct sum of k line bundles, and another rank $n+1-k$ bundle over Δ^-. We have already seen this decomposition when we identified F as being homeomorphic to a subbundle of the frame bundle $F(\kappa_{p_1,n}^{\otimes p_1} \oplus \cdots \oplus \kappa_{p_k,n}^{\otimes p_k} \oplus (\kappa_{p_1,n} \oplus \cdots \oplus \kappa_{p_k,n})^{\perp})$ in the previous paragraph. What remains is to check that the $U(1)^{\times k} \times U(n+1-k)$-action on F is the correct one.

Let the elements of $U(1)^{\times k} \times U(n+1-k)$ be denoted by $(\lambda_1, \dots, \lambda_k, h)$, where $\lambda_1, \dots, \lambda_k \in U(1)$ and $h \in U(n+1-k)$. We describe the action of $U(1)^{\times k} \times U(n+1-k)$ on F:

1. The action of the $U(n+1-k)$ factor is given by

$$(1, \dots, 1, h) \cdot [w_1, \dots, w_k, v_{k+1}, \dots, v_{n+1}] = [w_1, \dots, w_k, (v_{k+1}, \dots, v_{n+1}) \circ h^{-1}],$$

where the symbol $(v_{k+1}, \dots, v_{n+1}) \circ h^{-1}$ denotes the result of multiplying the matrix whose columns are $v_{k+1}, \dots, v_{n+1}$ by h^{-1} on the right. Geometrically, this action on a map $f \in F$ is the $U(n+1-k)$-action on the subspace of the codomain $\mathbf{C}^{n+1}$ orthogonal to $\mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_k$.

2. The action of the $U(1)$ factors is calculated explicitly as

$$(\lambda_1, \dots, \lambda_k, 1) \cdot F(\overline{w}_1, \cdots, \overline{w}_k, \dots, \mu_1, \dots, \mu_k) = F(\lambda_1 \overline{w}_1, \cdots, \lambda_k \overline{w}_k, \dots, \lambda_1 \mu_1, \dots, \lambda_k \mu_k).$$

Recalling the map (5.1) defined above, the $U(1)$-action corresponds to the action on each L_p^{2n+1} factor described by

$$\begin{aligned}
\lambda \cdot \{\, \nu \mu w \mid \nu^p = \mu^{-1},\ \nu \in S^1 \,\} &= \{\, \nu \mu w \mid \nu^p = \lambda^{-1} \mu^{-1},\ \nu \in S^1 \,\} \\
&= \{\, \lambda^{-1/p} \nu \mu v \mid \nu^p = \mu^{-1},\ \nu \in S^1 \,\},
\end{aligned}$$

where $\lambda^{-1/p}$ is any pth root of λ^{-1} (since multiplication by $\lambda^{-1/p}$ in the lens space does not depend on this choice). Hence, using the notation $[w_1, \dots, w_k, v_{k+1}, \dots, v_{n+1}]$ described a few paragraphs ago, for elements of F,

$$(\lambda_1, \dots, \lambda_k, 1) \cdot [w_1, \dots, w_1, v_{k+1}, \dots, v_{n+1}] = [\lambda_k^{-1/p_1} w_1, \dots, \lambda_1^{-1/p_k} w_k, v_{k+1}, \dots, v_{n+1}],$$

where $\lambda^{-1/p_i} w_i$ is well-defined because $v_i \in L^{2n+1}_{p_i}$. Geometrically, this action on a map $f \in F$ is the product of circle actions on each of the subspaces of the codomain $\mathbf{C}^{n+1}$ given by the spans $\mathbf{C}e_i$, $i = 1, \ldots, k$.

From the above description, the action of $U(1)^{\times k} \times U(n+1-k)$ on F is the same as the action on the subbundle of the frame bundle $F(\kappa_{p_1,n}^{\otimes p_1} \oplus \cdots \oplus \kappa_{p_k,n}^{\otimes p_k} \oplus (\kappa_{p_1,n} \oplus \cdots \oplus \kappa_{p_k,n})^{\perp})$ under our homeomorphism, and hence they are isomorphic as $U(1)^{\times k} \times U(n+1-k)$-bundles. Hence, we obtain the desired isomorphism

$$V\rho|_{\Delta^-} \cong \kappa_{p_1,n}^{\otimes p_1} \oplus \cdots \oplus \kappa_{p_k,n}^{\otimes p_k} \oplus (\kappa_{p_1,n} \oplus \cdots \oplus \kappa_{p_k,n})^{\perp}. \qquad \square$$

Remark 5.2.5 (Maximal anti-diagonal when $d = p^k$). We can define the maximal diagonal in the case where $d = p^k$ is a power of a prime p. This would be the subspace of $\mathcal{A}(n)_{p^k}$ given by

$$\Delta^-(n)/\Sigma_k = \{ [v_1 \circ \cdots \circ v_k] \mid v_1, \ldots, v_k \in S^{2n+1},\ v_i \perp v_j \text{ for all } i \neq j \}.$$

where all the atomic split polynomials are understood to have degree p. By Proposition 3.2.2 (Commutativity of atomic split polynomials), these maps all commute, but because every polynomial has the same degree, we are no longer able to identify any ordering. This justifies our notation $\Delta^-(n)/\Sigma_k$, because the anti-diagonal is homeomorphic to $\Delta^-_{\mathbf{C}P^n}/\Sigma_k$, where $\Delta^-_{\mathbf{C}P^n}$ is the anti-diagonal of the product $(\mathbf{C}P^n)^{\times k}$, and the symmetric group Σ_k acts on $(\mathbf{C}P^n)^{\times k}$ by permuting the factors. This action is free, and so there is a homotopy equivalence

$$\Delta^-(n)/\Sigma_k \simeq \Delta^-_{\mathbf{C}P^n}/\!\!/\Sigma_k$$

with the homotopy orbit space (c.f. Section 5.1). Taking the limit as $n \to \infty$, the *stable maximal anti-diagonal*

$$\Delta^-/\!\!/\Sigma_k := \varinjlim_n \Delta^-(n)/\Sigma_k$$

becomes homotopy equivalent to $(\mathbf{C}P^\infty)^{\times k}/\!\!/\Sigma_k$.

The importance of the space $(\mathbf{C}P^\infty)^{\times k}/\!\!/\Sigma_k$ comes from the following. Inside of the unitary group $U(k)$, there is the maximal torus T^k consisting of the diagonal matrices. Its normaliser in $U(k)$ is a semidirect product

$$N_k := N_{U(k)}(T^k) \cong T^k \rtimes \Sigma_k.$$

Applying [CG21, Lemma 2.2] to the split short exact sequence

$$1 \longrightarrow T^k \longrightarrow N_k \longrightarrow \Sigma_k \longrightarrow 1,$$

we obtain a homotopy equivalence of classifying spaces

$$BN_k \simeq BT^k/\!\!/\Sigma_k.$$

By functoriality of B and the fact that $\mathbf{C}P^\infty$ is a BS^1, the product $(\mathbf{C}P^\infty)^{\times k}$ is a model for the classifying space BT^k. What we conclude is that the stable maximal anti-diagonal for degree p^k is a model for the classifying space BN_k.

Conjecture 5.2.6. *In the notation of Remark 5.2.5, there is a homotopy equivalence $\mathcal{A}_{p^2} \simeq BN_2$, where $\mathcal{A}_{p^2}$ denotes the stable degree-p^2 $\mathcal{A}$-space (see Definition 3.5.2).*

5.2.2 The bundle $V\rho$ over the atomic $\mathcal{A}$-space

Recall the atomic split polynomial space $SP(n)^{\text{at}}$, the subspace of $SP(n)$ consisting of the atomic split polynomials and unitary maps:

$$SP(n)^{\text{at}} = \{ A \circ (v, d) \mid A \in U(n+1), v \in S^{2n+1}, d \in \mathbf{Z}_{>0} \}.$$

We remark that we do not need to write $A \circ (v, d) \circ B$ as we did in Definition 3.1.2 due to the existence of the normal form (see Definition 3.1.6). The atomic $\mathcal{A}$-space $\mathcal{A}(n)^{\text{at}}$ is the image of $SP(n)^{\text{at}}$ in $\mathcal{A}(n)$, and the spaces $SP(n)^{\text{at}}_d$ and $\mathcal{A}(n)^{\text{at}}_d$ denote the degree-d components of $SP(n)^{\text{at}}$ and $\mathcal{A}(n)^{\text{at}}$ respectively.

Remark 5.2.7. When d is a prime, $SP(n)^{\text{at}}_d$ and $SP(n)_d$ coincide, with the same holding for $\mathcal{A}(n)^{\text{at}}_d$ and $\mathcal{A}(n)_d$.

In light of Remark 3.4.7 (The atomic $\mathcal{A}$-space as complex projective space), $\mathcal{A}(n)^{\text{at}}_d \cong \mathbf{C}P^n$. Let $\gamma_n \to \mathbf{C}P^n$ denote the tautological line bundle over $\mathbf{C}P^n$, and $\gamma_n^{\perp} \to \mathbf{C}P^n$ its orthogonal complement. The homeomorphism $\mathcal{A}(n)^{\text{at}}_d \cong \mathbf{C}P^n$ gives us a tautological line bundle and its orthogonal complement over $\mathcal{A}(n)^{\text{at}}_d$, which we also denote by γ_n and $\gamma_n^{\perp}$ respectively by abuse of notation.

Theorem 5.2.8. *There is a vector bundle isomorphism* $V\rho|_{SP(n)^{\text{at}}_d} \cong \gamma_n^{\otimes d} \oplus \gamma_n^{\perp}$.

Proof. By Remark 5.2.7, this theorem is a corollary of Theorem 5.2.3 when $d = p$ is prime, for the maximal anti-diagonal of $\mathcal{A}(n)_p$ is the whole space $\mathcal{A}(n)_p$.

When d is not a prime, the proof of Theorem 5.2.3 generalises in the atomic case due to the following observation. Restricted to over the atomic $\mathcal{A}$-space, the structure of the split polynomials simplifies dramatically: a map $f \in SP(n)^{\text{at}}_d$ has a normal form consisting of a single atomic spit polynomial

$$f = A \circ (v, d),$$

for some $A \in U(n+1)$ and $v \in S^{2n+1}$. Hence, the proof for Theorem 5.2.3 specialised to $\mathcal{A}(n)_p$ for p prime applies to $\mathcal{A}(n)^{\text{at}}_d$ after replacing all instances of p with d. $\square$

6 The cohomology of the $\mathcal{A}$-space

This chapter is entirely dedicated to computing the cohomology of the $\mathcal{A}$-space in various degrees, and especially for the two cases: when $d = p^2$ is a square of a prime, and when $d = pq$ the product of two distinct primes. We also prove some results about the cohomology of the $\mathcal{A}$-space stably.

Notation 6.0.1. Throughout this chapter, the cohomology of a space X, denoted by $H^i(X)$, is implicitly assumed to have $\mathbf{Z}$-coefficients unless specified otherwise.

6.1 The case of $d = p$, p prime

We briefly state the case when $d = p$ is a prime for completeness.

Theorem 6.1.1 (The cohomology of $\mathcal{A}(n)_p$). *The cohomology ring of $\mathcal{A}(n)_p$ is the truncated polynomial ring $\mathbf{Z}[c_1]/(c_1^{n+1})$ [Hat01, Theorem 3.19].*

Proof. In Section 3.4.3, we have seen that $\mathcal{A}(n)_p \cong \mathbf{C}P^n$. Therefore the cohomology of $\mathcal{A}(n)_p$ is isomorphic to the cohomology of $\mathbf{C}P^n$. □

6.2 The case of $d = p^2$, p prime

This section deals with the cohomology of the degree-p^2 $\mathcal{A}$-space. Here is our main result.

Theorem 6.2.1 (The rational cohomology of $\mathcal{A}(n)_{p^2}$). *The rational cohomology groups of $\mathcal{A}(n)_{p^2}$ are given by*

$$H^i(\mathcal{A}(n)_{p^2}; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}^{\oplus r_j}, & \text{for } i = 2j, \text{ where } 0 \leqslant j \leqslant n-1, \\ \mathbf{Q}^{\oplus (r_j+1)}, & \text{for } i = 2j, \text{ where } n \leqslant j \leqslant 2n-1, \\ \mathbf{Q}, & \text{for } i = 2j, \text{ where } j = 2n, \\ 0, & \text{otherwise,} \end{cases}$$

where r_j, $j \in \mathbf{Z}$, are the coefficients of the q-binomial coefficient

$$\sum_{j=-\infty}^{+\infty} r_j q^j = \binom{n+1}{2}_q := \frac{(1-q^{n+1})(1-q^n)}{(1-q)(1-q^2)} \in \mathbf{Z}[q]. \tag{6.1}$$

We will build up to this theorem with a sequence of lemmas computing the cohomology of various subspaces of $\mathcal{A}(n)_{p^2}$.

In Section 3.4.4, we described a model for the degree-p^2 component of the $\mathcal{A}$-space. In particular, by Corollary 3.4.1, $\mathcal{A}(n)_{p^2}$ is homeomorphic to the quotient of twisted balanced product $\widetilde{\mathcal{A}}(n)_{p^2} = \mathbf{C}P^n \,\widetilde{\times}_{S^1}\, L_{p-1}^{2n+1}$ (see equation (3.2)) by the equivalence relation $\sim_{p^2}$. We analyse the stratification structure of $\mathcal{A}(n)_{p^2}$ to create a Mayer-Vietoris sequence.

Claim 6.2.2. *The anti-diagonal Δ^- is a deformation retract of the complement of the diagonal $\widetilde{\mathcal{A}}(n)_{p^2} \setminus \Delta$ via the Gram-Schmidt process.*

Proof. For $[v, w] \in \mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{p-1}$ such that $v \nparallel w$, v admits an orthogonal decomposition

$$v = v_\parallel + v_\perp, \quad \text{where} \quad v_\parallel \in \mathbf{C}w,\ v_\perp \in (\mathbf{C}w)^\perp \setminus \{0\}.$$

Define the deformation retraction

$$\begin{array}{rccc} r_t: & \widetilde{\mathcal{A}}(n)_{p^2} \setminus \Delta & \longrightarrow & \widetilde{\mathcal{A}}(n)_{p^2} \setminus \Delta \\ & [v, w] & \longmapsto & \left[\dfrac{(1-t)v_\parallel + v_\perp}{\|(1-t)v_\parallel + v_\perp\|}, w\right]. \end{array}$$

To see that r_t is well-defined, we check the following two things:

1. Let Δ' denote the preimage of Δ in $\mathbf{C}P^n \times L^{2n+1}_{p-1}$. Consider the map on the product

$$\begin{array}{ccc} (\mathbf{C}P^n \times L^{2n+1}_{p-1}) \setminus \Delta' & \longrightarrow & (\mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{p-1}) \setminus \Delta \\ [v, w] & \longmapsto & \left[\dfrac{(1-t)v_\parallel + v_\perp}{\|(1-t)v_\parallel + v_\perp\|}, w\right]. \end{array}$$

   To see that this map is well-defined, consider the pair $(\lambda v, \mu w) \in S^{2n+1} \times S^{2n+1}$ where $\lambda, \mu \in S^1$ with $\mu^{p-1} = 1$. Since $\mathbf{C}w = \mathbf{C}\mu w$, λv has orthogonal decomposition

$$\lambda v = \lambda v_\parallel + \lambda v_\perp, \qquad \lambda v_\parallel \in \mathbf{C}\mu w,\ \lambda v_\perp \in (\mathbf{C}\mu w)^\perp \setminus \{0\}.$$

   From this, we see that

$$\left[\frac{(1-t)\lambda v_\parallel + \lambda v_\perp}{\|(1-t)\lambda v_\parallel + \lambda v_\perp\|}, \mu w\right] = \left[\frac{(1-t)v_\parallel + v_\perp}{\|(1-t)v_\parallel + v_\perp\|}, w\right].$$

2. To see if we can then descend to the quotient $\mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{p-1} = S^1 \backslash (\mathbf{C}P^n \times L^{2n+1}_{p-1})$, now consider the pair $(A_w^{\mu^{1-p}} v, \mu w)$ for some $\mu \in S^1$. We still have that $\mathbf{C}w = \mathbf{C}\mu w$, but now $A_w^{\mu^{1-p}} v$ has orthogonal decomposition

$$A_w^{\mu^{1-p}} v = A_w^{\mu^{1-p}} (v_\parallel + v_\perp) = \mu^{1-p} v_\parallel + v_\perp, \quad \mu^{1-p} v_\parallel \in \mathbf{C}\mu w,\ v_\perp \in (\mathbf{C}\mu w)^\perp \setminus \{0\}.$$

   Indeed,

$$\left[\frac{(1-t)\mu^{1-p} v_\parallel + v_\perp}{\|(1-t)\mu^{1-p} v_\parallel + v_\perp\|}, \mu w\right] = \left[\frac{A_w^{\mu^{1-p}}((1-t)v_\parallel + v_\perp)}{\|(1-t)v_\parallel + v_\perp\|}, \mu w\right] = \left[\frac{(1-t)v_\parallel + v_\perp}{\|(1-t)v_\parallel + v_\perp\|}, w\right].$$

So r_t is well-defined. This shows that Δ^- is a deformation retract of $\widetilde{\mathcal{A}}(n)_{p^2} \setminus \Delta$. □

Manifold structure of $\widetilde{\mathcal{A}}(n)_{p^2}$. Recall that $\widetilde{\mathcal{A}}(n)_{p^2} = \mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{p-1}$ is the quotient $S^1 \backslash (\mathbf{C}P^n \times L^{2n+1}_{p-1})$ of the free S^1-action on $\mathbf{C}P^n \times L^{2n+1}_{p-1}$. The action is automatically proper by compactness of S^1. So $\widetilde{\mathcal{A}}(n)_{p^2}$ is a smooth manifold with (real) dimension

$$\dim_{\mathbf{R}} \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{p-1} = \dim_{\mathbf{R}} \mathbf{C}P^n + \dim_{\mathbf{R}} L^{2n+1}_{p-1} - \dim_{\mathbf{R}} S^1 = 4n.$$

The subspace Δ is a smooth submanifold of $\widetilde{\mathcal{A}}(n)_{p^2}$ of codimension $2n$. It has a tubular neighbourhood N diffeomorphic to the disc bundle $D\nu(\Delta)$ of the normal bundle $\nu(\Delta \hookrightarrow \widetilde{\mathcal{A}}(n)_{p^2})$. The radius of N can be taken to be small enough such that it is disjoint from Δ^-. So we identify N with its image in the quotient $\mathcal{A}(n)_{p^2}$.

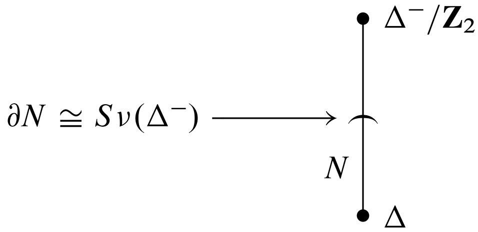

Figure 6.1: Schematic diagram of $\mathcal{A}(n)_{p^2}$.

Constructing the Mayer-Vietoris sequence. Consider the cover of $\mathcal{A}(n)_{p^2}$ consisting of N and the image of $\widetilde{\mathcal{A}}(n)_{p^2} \setminus \Delta$. We note the following:

- As a tubular neighbourhood, N deformation retracts onto $\Delta \cong \mathbf{C}P^n$.
- $\widetilde{\mathcal{A}}(n)_{p^2} \setminus \Delta$ deformation retracts onto Δ^- in $\widetilde{\mathcal{A}}(n)_{p^2}$ by Claim 6.2.2. So its image deformation retracts onto $\Delta^-/\mathbf{Z}_2$ in $\mathcal{A}(n)_{p^2}$.
- By choosing N small enough, their intersection can be made to deformation retract onto the boundary of N. By construction of the tubular neighbourhood, ∂N is diffeomorphic to the sphere bundle $S\nu(\Delta)$. Since Δ has codimension $2n$, the normal bundle $\nu(\Delta)$ has (real) rank $2n$, and therefore its sphere bundle $S\nu(\Delta)$ is a S^{2n-1}-bundle.

The resulting Mayer-Vietoris sequence is the following:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{A}(n)_{p^2}) \arrow[r] & H^0(\Delta) \oplus H^0(\Delta^-/\mathbf{Z}_2) \arrow[r] & H^0(S\nu(\Delta)) \arrow[dll, out=-10, in=170] \\
& H^1(\mathcal{A}(n)_{p^2}) \arrow[r] & H^1(\Delta) \oplus H^1(\Delta^-/\mathbf{Z}_2) \arrow[r] & H^1(S\nu(\Delta)) \arrow[dll, out=-10, in=170] \\
& H^2(\mathcal{A}(n)_{p^2}) \arrow[r] & H^2(\Delta) \oplus H^2(\Delta^-/\mathbf{Z}_2) \arrow[r] & H^2(S\nu(\Delta)) \arrow[dll, out=-10, in=170] \\
& \cdots .
\end{tikzcd} \tag{6.2}$$

The remaining part of this section will be spent calculating the groups in this sequence.

Calculating the Mayer-Vietoris sequence

Lemma 6.2.3. *The cohomology of Δ is isomorphic to the cohomology of $\mathbf{C}P^n$, given by*

$$H^i(\Delta) \cong \begin{cases} \mathbf{Z}, & \text{for } i \text{ even and } 0 \leqslant i \leqslant 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Because Δ is identified with the diagonal $\Delta_{\mathbf{C}P^n} \subseteq \mathbf{C}P^n \times \mathbf{C}P^n$, we have a homeomorphism $\Delta \cong \mathbf{C}P^n$. The cohomology ring of $\mathbf{C}P^n$ is the truncated polynomial ring $\mathbf{Z}[c_1]/(c_1^{n+1})$ where $c_1 = c_1(\gamma_n) \in H^2(\mathbf{C}P^n)$ is the 1st Chern class of the tautological line bundle $\gamma_n \to \mathbf{C}P^n$ [MS74, Theorem 14.4]. $\square$

Lemma 6.2.4. *The cohomology of $\Delta^-/\mathbf{Z}_2$ is*

$$H^i(\Delta^-/\mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}^{\oplus r_j}, & \text{for } i = 2j, \text{ where } j = 0, \\ \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}}, & \text{for } i = 2j, \text{ where } j = 1, 2, \dots, 2n-2, \\ \mathbf{Z}_2^{\oplus r_{j-1}}, & \text{for } i = 2j, \text{ where } j = 2n-1, \\ 0, & \text{otherwise}, \end{cases}$$

where r_j are integers defined in equation (6.1).

There is a fibre bundle $\mathbf{C}P^1 \to \Delta^- \to G_2(\mathbf{C}^{n+1})$ where the projection map $\Delta^- \to G_2(\mathbf{C}^{n+1})$ given by $[v, w] \mapsto \mathrm{span}_{\mathbf{C}}\{v, w\}$. Quotienting by the $\mathbf{Z}_2$ action on the diagonal gives rise to the fibre bundle $\mathbf{R}P^2 \to \Delta^-/\mathbf{Z}_2 \to G_2(\mathbf{C}^{n+1})$. These bundles fit into the following commutative diagram:

$$\begin{tikzcd} \mathbf{C}P^1 \arrow[r] \arrow[d, hook] & \mathbf{R}P^2 \arrow[d, hook] \\ \Delta^- \arrow[r] \arrow[d, two heads] & \Delta^-/\mathbf{Z}_2 \arrow[d, two heads] \\ G_2(\mathbf{C}^{n+1}) \arrow[r, equal] & G_2(\mathbf{C}^{n+1}). \end{tikzcd}$$

The cohomology of $\Delta^-/\mathbf{Z}_2$ can be computed using the Serre spectral sequence applied to the fibre bundle above. Because the base space $G_2(\mathbf{C}^{n+1})$ is simply connected, the spectral sequence has untwisted coefficients.

Lemma 6.2.5. *The cohomology groups of $G_2(\mathbf{C}^{n+1})$ are*

$$H^i(G_2(\mathbf{C}^{n+1})) \cong \begin{cases} \mathbf{Z}^{\oplus r_j}, & \text{for } i = 2j, \text{ where } 0 \leqslant j \leqslant 2n-2, \\ 0, & \text{otherwise}, \end{cases}$$

where r_j are integers defined in equation (6.1).

Remark 6.2.6. As a ring, the cohomology of $G_2(\mathbf{C}^{n+1})$ is a quotient of the polynomial ring $\mathbf{Z}[c_1, c_2]$ by an ideal I [MS74, Theorem 14.5]. The two generators correspond to $c_1 = c_1(\omega^2) \in H^2(G_2(\mathbf{C}^{n+1}))$ and $c_2 = c_2(\omega^2) \in H^4(G_2(\mathbf{C}^{n+1}))$, the first two Chern classes of the tautological rank 2 bundle $\omega^2 \to G_2(\mathbf{C}^{n+1})$. The ideal I is defined such that the Cartan formula [MS74, Formula 14.7]

$$1 = c(\omega^2 \oplus (\omega^2)^\perp) = c(\omega^2)\, c((\omega^2)^\perp)$$

holds in the quotient $\mathbf{Z}[c_1, c_2]/I$.

Proof of Lemma 6.2.5. The cohomology groups of the complex Grassmannian are calculated as follows. To each plane $\Pi \in G_2(\mathbf{C}^{n+1})$ we associate the reduced row echelon form of a matrix $A_\Pi \in \mathrm{Mat}_{2\times(n+1)}(\mathbf{C})$ whose rows span Π. The map $\Pi \mapsto \operatorname{rref} A_\Pi$ is well-defined by [Hat17, Section 1.2], and we obtain a CW-structure on $G_2(\mathbf{C}^{n+1})$ with one cell $e(\sigma)$ of dimension $2((\sigma_1 - 1) + (\sigma_2 - 2))$ for each *Schubert symbol* $\sigma = (\sigma_1, \sigma_2)$, $1 \leqslant \sigma_1 < \sigma_2 \leqslant n + 1$. For example:

- When $n = 1$, there is only one Schubert symbol $(1, 2)$, giving rise to a cell of dimension 0.
- When $n = 2$, there are three Schubert symbols $(1, 2)$, $(1, 3)$, $(2, 3)$, giving rise to cells of dimension 0, 2, 4.
- When $n = 3$, there are six Schubert symbols $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, $(3, 4)$, giving rise to cells of dimension 0, 2, 4, 4, 6, 8.
- When $n = 4$, there are ten Schubert symbols $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, $(4, 5)$, giving rise to cells of dimension 0, 2, 4, 6, 4, 6, 8, 8, 10, 12.
- In general, there are $\binom{n+1}{2}$ Schubert symbols, giving rise to r_j cells of dimension $2j$ for each $0 \leqslant j \leqslant 2n - 2$.

Because all cells are of even dimension, cellular cohomology can be used to see that the cohomology groups are free with rank equal to the number of cells of the corresponding dimension.

To calculate the ring structure, a proof is found in [MS74, Theorem 14.5]. ☐

We now return to the spectral sequence.

Proof of Lemma 6.2.4. The E_2 page of the Serre spectral sequence for the fibre bundle $\mathbf{R}P^2 \to \Delta^-/\mathbf{Z}_2 \to G_2(\mathbf{C}^{n+1})$ is given by:

$$
\begin{array}{c|cccccc}
t & & & & & & \\
2 & \mathbf{Z}_2^{\oplus r_0} & \mathbf{Z}_2^{\oplus r_1} & \mathbf{Z}_2^{\oplus r_2} & \cdots & \mathbf{Z}_2^{\oplus r_{2n-3}} & \mathbf{Z}_2^{\oplus r_{2n-2}} \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \mathbf{Z}^{\oplus r_0} & \mathbf{Z}^{\oplus r_1} & \mathbf{Z}^{\oplus r_2} & \cdots & \mathbf{Z}^{\oplus r_{2n-3}} & \mathbf{Z}^{\oplus r_{2n-2}} \\
\hline
 & 0 & 2 & 4 & \cdots & 4n-6 & 4n-4 \quad s
\end{array}
$$

There are no non-zero differentials possible because all non-zero groups are concentrated in the even dimensions. Because $\mathbf{Z}^{\oplus r_j}$ is free for all j, there are no extension problems. So the group structure of $H^i(\Delta^-/\mathbf{Z}_2)$ coincides with its associated graded, and we conclude that

$$
H^i(\Delta^-/\mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}^{\oplus r_j}, & \text{for } i = 2j, \text{ where } j = 0, \\ \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}}, & \text{for } i = 2j, \text{ where } j = 1, 2, \ldots, 2n-2, \\ \mathbf{Z}_2^{\oplus r_{j-1}}, & \text{for } i = 2j, \text{ where } j = 2n-1, \\ 0, & \text{otherwise,} \end{cases}
$$

as desired. ☐

Lemma 6.2.7. *The cohomology of $S\nu(\Delta)$ is*

$$H^i(S\nu(\Delta)) \cong \begin{cases} \mathbf{Z}, & \textit{for } i = 0, 2, \ldots, 2n-2, \\ \mathbf{Z}_e, & \textit{for } i = 2n, \\ \mathbf{Z}, & \textit{for } i = 2n+1, 2n+3, \ldots, 4n-1, \\ 0, & \textit{otherwise,} \end{cases}$$

where $e \in \mathbf{Z}$ corresponds to the Euler class of $\nu(\Delta)$ under the isomorphism $H^{2n}(\Delta) \cong \mathbf{Z}$.

Proof. We have the sphere bundle $S^{2n-1} \to S\nu(\Delta) \to \Delta$, to which we apply the Serre spectral sequence. Since $\Delta \cong \mathbf{C}P^n$ is simply connected, the spectral sequence has untwisted coefficients. The E_2 page is given by:

$$\begin{tikzcd}[row sep=small]
t & & & & & & \\
2n-1 & \mathbf{Z} \arrow[ddrrrrr, "e\cup"] & \mathbf{Z} & \mathbf{Z} & \cdots & \mathbf{Z} & \mathbf{Z} \\
\vdots & & & & & & \\
0 & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \cdots & \mathbf{Z} & \mathbf{Z} & s \\
 & 0 & 2 & 4 & \cdots & 2n-2 & 2n
\end{tikzcd}$$

The only possible non-zero differential shown in the diagram above is $d_{2n}^{0,2n-1} : E_{2n}^{0,2n-1} \to E_{2n}^{2n,0}$ on the E_{2n} page. It is the Gysin homomorphism, given by cupping with the Euler class $e = e(\nu(\Delta)) \in H^{2n}(\Delta)$ [MS74, Theorem 12.2]. The resulting E_∞ page is:

$$\begin{array}{c|cccccc}
t & & & & & & \\
2n-1 & 0 & \mathbf{Z} & \mathbf{Z} & \cdots & \mathbf{Z} & \mathbf{Z} \\
\vdots & & & & & & \\
0 & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \cdots & \mathbf{Z} & \mathbf{Z}_e \\
\hline
 & 0 & 2 & 4 & \cdots & 2n-2 & 2n \quad s
\end{array} \tag{6.3}$$

Each group $\mathbf{Z}$ along the bottom row is free, so we have no extension problems. Hence, the group structure of $H^i(S\nu(\Delta))$ coincides with its associated graded, from which we yield the desired result. $\square$

Remark 6.2.8 (Boundary homomorphisms). By [McC01, Theorem 5.9], the boundary terms $E_\infty^{i,0}$ of the spectral sequence above (6.3) are the images of the *boundary homomorphisms* $\operatorname{im}(H^i(\Delta) \to H^i(S\nu(\Delta)))$ induced by the projection map $S\nu(\Delta) \to \Delta$. These boundary homomorphisms correspond to the maps $H^i(\mathbf{C}P^n) \to H^i(S\nu(\Delta))$ in the Mayer-Vietoris sequence for $\mathcal{A}(n)_{p^2}$. Since $H^i(S\nu(\Delta)) = E_\infty^{i,0}$ for $0 \leqslant i \leqslant 2n$, $H^i(\mathbf{C}P^n) \to H^i(S\nu(\Delta))$ is surjective in this range.

Proof of Theorem 6.2.1. The above lemmas allow us to fill in the groups of the Mayer-Vietoris sequence (6.2). We find that the sequence has two distinct portions:

1. In dimensions $0 \leqslant i \leqslant 2n$, we have

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[dll, out=-10, in=170] \\
H^{2j}(\mathcal{A}(n)_{p^2}) \arrow[r, hook] & \mathbf{Z} \oplus \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}} \arrow[r, two heads] & H^{2j}(S\nu(\Delta)) \arrow[dll, out=-10, in=170, "0"'] \\
H^{2j+1}(\mathcal{A}(n)_{p^2}) \arrow[r, hook] & 0 \arrow[r] & \cdots.
\end{tikzcd}$$

The boundary homomorphism (see Remark 6.2.8) gives surjectivity onto $H^{2j}(S\nu(\Delta))$, which forces $H^{2j+1}(\mathcal{A}(n)_{p^2}) = 0$. This yields short exact sequences

$$0 \longrightarrow H^{2j}(\mathcal{A}(n)_{p^2}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}} \longrightarrow H^{2j}(S\nu(\Delta)) \longrightarrow 0$$

for all $0 \leqslant j \leqslant n$.

When $j < n$, $H^{2j}(S\nu(\Delta)) \cong \mathbf{Z}$ and so the short exact sequence is

$$0 \longrightarrow H^{2j}(\mathcal{A}(n)_{p^2}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}} \longrightarrow \mathbf{Z} \longrightarrow 0.$$

Because $\mathbf{Z}$ is free, this short exact sequence is split. By the structure theorem of finitely generated abelian groups, $H^{2j}(\mathcal{A}(n)_{p^2}) \cong \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}}$. Tensoring with $\mathbf{Q}$, we find that $H^{2j}(\mathcal{A}(n)_{p^2}; \mathbf{Q}) \cong \mathbf{Q}^{\oplus r_j}$.

In dimension $2n$, $H^{2n}(S\nu(\Delta)) \cong \mathbf{Z}_e$ and so the short exact sequence is

$$0 \longrightarrow H^{2n}(\mathcal{A}(n)_{p^2}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}^{\oplus r_n} \oplus \mathbf{Z}_2^{\oplus r_{n-1}} \longrightarrow \mathbf{Z}_e \longrightarrow 0.$$

Such a short exact sequence is not split in general. We can make the calculation rationally by tensoring with $\mathbf{Q}$ to find that $H^{2n}(\mathcal{A}(n)_{p^2}; \mathbf{Q}) \cong \mathbf{Q}^{\oplus(r_n+1)}$.

2. In dimensions $2n + 1 \leqslant i \leqslant 4n$, we have

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbf{Z} \arrow[dll, hook, out=-10, in=170] \\
H^{2j}(\mathcal{A}(n)_{p^2}) \arrow[r, two heads] & \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}} \arrow[r] & 0 \arrow[dll, out=-10, in=170] \\
H^{2j+1}(\mathcal{A}(n)_{p^2}) \arrow[r] & 0 \arrow[r] & \cdots.
\end{tikzcd}$$

Each $H^{2j+1}(\mathcal{A}(n)_{p^2})$ is sandwiched between two 0s, forcing $H^{2j+1}(\mathcal{A}(n)_{p^2}) = 0$. We also have short exact sequences

$$0 \longrightarrow \mathbf{Z} \longrightarrow H^{2j}(\mathcal{A}(n)_{p^2}) \longrightarrow \mathbf{Z}^{\oplus r_j} \oplus \mathbf{Z}_2^{\oplus r_{j-1}} \longrightarrow 0.$$

for all $n+1 \leqslant j \leqslant 2n$. Tensoring with $\mathbf{Q}$, we find that $H^{2j}(\mathcal{A}(n)_{p^2}; \mathbf{Q}) \cong \mathbf{Q}^{\oplus(r_j+1)}$.

3. All terms are zero in dimensions $i > 4n$.

We restate the result that we have just calculated:

$$H^i(\mathcal{A}(n)_{p^2}; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}^{\oplus r_j}, & \text{for } i = 2j, \text{ where } 0 \leqslant j \leqslant n-1, \\ \mathbf{Q}^{\oplus(r_j+1)}, & \text{for } i = 2j, \text{ where } n \leqslant j \leqslant 2n-1, \\ \mathbf{Q}, & \text{for } i = 2j, \text{ where } j = 2n, \\ 0, & \text{otherwise.} \end{cases}$$

□

6.3 The stable cohomology of $\mathcal{A}_{p^2}$

In the proof for Theorem 6.2.1 (The rational cohomology of $\mathcal{A}(n)_{p^2}$), we were able to compute $H^{2j}(\mathcal{A}(n)_{p^2})$ integrally for $j < n$. In this section, we extend this result to the stable $\mathcal{A}$-space of degree p^2.

Theorem 6.3.1 (Stable cohomology of $\mathcal{A}_{p^2}$). *The inclusion $\mathcal{A}(n)_{p^2} \hookrightarrow \mathcal{A}(n+1)_{p^2}$ induces isomorphisms*

$$H^i(\mathcal{A}(n)_{p^2}) \cong H^i(\mathcal{A}(n+1)_{p^2})$$

for all $i \leqslant 2n-2$. Furthermore, the integral cohomology groups of $\mathcal{A}_{p^2}$ are given by

$$H^i(\mathcal{A}_{p^2}) \cong \begin{cases} \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus j}, & \text{for } i = 4j, \\ \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus(j+1)}, & \text{for } i = 4j+2, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6.3.2 (Rational stable cohomology of $\mathcal{A}_{p^2}$). *The rational cohomology groups of $\mathcal{A}_{p^2}$ are given by*

$$H^i(\mathcal{A}_{p^2}; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}^{\oplus(j+1)}, & \text{for } i = 2j, \\ 0, & \text{otherwise.} \end{cases}$$

Notation 6.3.3. For the rest of this section, we denote the diagonal and anti-diagonal of $\mathcal{A}(n)_{p^2}$ by $\Delta(n)$ and $\Delta^-(n)/\mathbf{Z}_2$ respectively.

Lemma 6.3.4. *The inclusions $\Delta(n) \hookrightarrow \Delta(n+1)$ induce isomorphisms*

$$H^i(\Delta(n+1)) \cong H^i(\Delta(n))$$

for all $i \leqslant 2n$.

Proof. Let $\Delta(n) \subseteq \mathbf{C}P^n \times \mathbf{C}P^n$ be the diagonal of the product. We have the homeomorphisms $\Delta(n) \cong \Delta(n) \cong \mathbf{C}P^n$ which fit into the commutative diagram

$$\begin{tikzcd}
\mathbf{C}P^n \arrow[r, hook] & \mathbf{C}P^{n+1} \\
\Delta(n) \arrow[r, hook] \arrow[u, "\cong"] & \Delta(n+1) \arrow[u, "\cong"] \\
\Delta(n) \arrow[r, hook] \arrow[u, "\cong"] \arrow[d, hook] & \Delta(n+1) \arrow[u, "\cong"] \arrow[d, hook] \\
\mathcal{A}(n)_{p^2} \arrow[r, hook] & \mathcal{A}(n+1)_{p^2}.
\end{tikzcd}$$

The inclusions $\mathbf{C}P^n \hookrightarrow \mathbf{C}P^{n+1}$ induce isomorphisms

$$H^i(\mathbf{C}P^{n+1}) \cong H^i(\mathbf{C}P^n)$$

for all $i \leqslant 2n$. In particular, the cohomology ring of $\mathbf{C}P^\infty$ is the polynomial ring $\mathbf{Z}[c_1]$ where $c_1 = c_1(\gamma) \in H^2(\mathbf{C}P^\infty)$ is the 1st Chern class of the tautological line bundle $\gamma \to \mathbf{C}P^\infty$. Hence, the inclusions $\Delta(n) \hookrightarrow \Delta(n+1)$ also induce isomorphisms

$$H^i(\Delta(n+1)) \cong H^i(\Delta(n))$$

for all $i \leqslant 2n$. □

Lemma 6.3.5. *The inclusion $\Delta^-(n)/\mathbf{Z}_2 \hookrightarrow \Delta^-(n+1)/\mathbf{Z}_2$ induces isomorphisms*

$$H^i(\Delta^-(n+1)/\mathbf{Z}_2) \cong H^i(\Delta^-(n)/\mathbf{Z}_2)$$

for all $i \leqslant 2n$.

Proof. The inclusion $\mathbf{C}^{n+1} \hookrightarrow \mathbf{C}^{n+2}$ induces inclusions $\Delta^-(n)/\mathbf{Z}_2 \hookrightarrow \Delta^-(n+1)/\mathbf{Z}_2$ and $G_2(\mathbf{C}^{n+1}) \hookrightarrow G_2(\mathbf{C}^{n+2})$. These inclusions fit into the fibre bundle commutative diagram

$$\begin{tikzcd}
\mathbf{R}P^2 \arrow[r, equal] \arrow[d, hook] & \mathbf{R}P^2 \arrow[d, hook] \\
\Delta^-(n)/\mathbf{Z}_2 \arrow[r, hook] \arrow[d, two heads] & \Delta^-(n+1)/\mathbf{Z}_2 \arrow[d, two heads] \\
G_2(\mathbf{C}^{n+1}) \arrow[r, hook] & G_2(\mathbf{C}^{n+2}).
\end{tikzcd}$$

Top horizontal map induces an isomorphism on all cohomology groups, while the bottom horizontal map induces isomorphisms

$$H^i(G_2(\mathbf{C}^{n+2})) \cong H^i(G_2(\mathbf{C}^{n+1}))$$

for all $i \leqslant 2n$. Denote the E_2 page of the Serre spectral sequence for $\mathbf{R}P^2 \to \Delta^-(n)/\mathbf{Z}_2 \to G_2(\mathbf{C}^{n+1})$ by $E_2(n)$, and for $\mathbf{R}P^2 \to \Delta^-(n+1)/\mathbf{Z}_2 \to G_2(\mathbf{C}^{n+2})$ by $E_2(n+1)$. We have by naturality of the Serre spectral sequence induced natural isomorphisms

$$E_2^{s,t}(n+1) \cong E_2^{s,t}(n)$$

for all $p+q \leqslant 2n$. From the calculation made in the proof of Lemma 6.2.4, the spectral sequences degenerate on the E_2 page, and therefore the inclusion $\Delta^-(n)/\mathbf{Z}_2 \hookrightarrow \Delta^-(n+1)/\mathbf{Z}_2$ induces isomorphisms

$$H^i(\Delta^-(n+1)/\mathbf{Z}_2) \cong H^i(\Delta^-(n)/\mathbf{Z}_2)$$

for all $i \leqslant 2n$. □

Lemma 6.3.6. *The inclusion $S\nu(\Delta(n)) \hookrightarrow S\nu(\Delta(n+1))$ induces isomorphisms*

$$H^i(\Delta(n+1)) \cong H^i(\Delta(n))$$

for all $i \leqslant 2n-2$.

Proof. The argument proceeds as in the proof of Lemma 6.3.5. There are induced inclusions $\Delta(n) \hookrightarrow \Delta(n+1)$ and $S\nu(\Delta(n)) \hookrightarrow S\nu(\Delta(n+1))$, yielding the fibre bundle commutative diagram

$$\begin{tikzcd}
S^{2n-1} \arrow[r, hook] \arrow[d, hook] & S^{2n+1} \arrow[d, hook] \\
S\nu(\Delta(n)) \arrow[r, hook] \arrow[d, two heads] & S\nu(\Delta(n+1)) \arrow[d, two heads] \\
\Delta(n) \arrow[r, hook] & \Delta(n+1).
\end{tikzcd}$$

Recalling that $\Delta(n) \cong \mathbf{C}P^n$, the inclusions induce isomorphisms on the base

$$H^i(\Delta(n+1)) \cong H^i(\Delta(n))$$

for all $i \leqslant 2n$, and on the fibre

$$H^i(S^{2n+1}) \cong H^i(S^{2n-1})$$

for all $i \leqslant 2n-2$. The only non-zero differentials in the Serre spectral sequences for the above bundles are the respective Gysin homomorphisms as seen in the proof of Lemma 6.2.7. These homomorphisms do not hit anything in degree $i \leqslant 2n-2$. So by naturality, the inclusion $S\nu(\Delta(n)) \hookrightarrow S\nu(\Delta(n+1))$ induces isomorphisms

$$H^i(S\nu(\Delta(n+1))) \cong H^i(S\nu(\Delta(n)))$$

for all $i \leqslant 2n-2$. □

The above lemmas provide us with a range in which the cohomology groups of the Mayer-Vietoris sequence (6.2) are stable. Exploiting this, we return to Theorem 6.3.1.

Proof of Theorem 6.3.1. Considering the Mayer-Vietoris sequences for $\mathcal{A}(n)_{p^2}$ and $\mathcal{A}(n+1)_{p^2}$, we have by naturality a commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{i-1}(\mathbf{C}P^{n+1}) \oplus H^{i-1}(\Delta^-(n+1)/\mathbf{Z}_2) \arrow[r, "\cong"] \arrow[d] & H^{i-1}(\mathbf{C}P^{n}) \oplus H^{i-1}(\Delta^-(n)/\mathbf{Z}_2) \arrow[d] \\
H^{i-1}(S\nu(\Delta(n+1))) \arrow[r, "\cong"] \arrow[d] & H^{i-1}(S\nu(\Delta(n))) \arrow[d] \\
H^{i}(\mathcal{A}(n+1)_{p^2}) \arrow[r] \arrow[d] & H^{i}(\mathcal{A}(n)_{p^2}) \arrow[d] \\
H^{i}(\mathbf{C}P^{n+1}) \oplus H^{i}(\Delta^-(n+1)/\mathbf{Z}_2) \arrow[r, "\cong"] \arrow[d] & H^{i}(\mathbf{C}P^{n}) \oplus H^{i}(\Delta^-(n)/\mathbf{Z}_2) \arrow[d] \\
H^{i}(S\nu(\Delta(n+1))) \arrow[r, "\cong"] \arrow[d] & H^{i}(S\nu(\Delta(n))), \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

where 4 out of 5 of the horizontal maps are isomorphisms for all $i \leqslant 2n-2$. Hence, by the 5-lemma, $H^i(\mathcal{A}(n+1)_{p^2}) \to H^i(\mathcal{A}(n)_{p^2})$ is an isomorphism for all $i \leqslant 2n-2$.

From the proof of Theorem 6.2.1, we have for each positive integer n, the integral cohomology of $\mathcal{A}(n)_{p^2}$ in the stable range $i \leqslant 2n-2$ is given by

$$H^i(\mathcal{A}(n)_{p^2}) \cong \begin{cases} \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus j}, & \text{for } i = 4j, \\ \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus(j+1)}, & \text{for } i = 4j+2. \end{cases}$$

Hence, by functoriality of cohomology and taking the limit as $n \to \infty$, we find that

$$H^i(\mathcal{A}_{p^2}) \cong \begin{cases} \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus j}, & \text{for } i = 4j, \\ \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus(j+1)}, & \text{for } i = 4j+2, \\ 0, & \text{otherwise,} \end{cases}$$

as desired. □

6.4 The case of $d = pq$, p, q distinct primes

This section deals with the cohomology of the degree-pq $\mathcal{A}$-space. Here is our main result.

Theorem 6.4.1 (The rational cohomology of $\mathcal{A}(n)_{pq}$). *The rational cohomology groups of $\mathcal{A}(n)_{pq}$ are given by*

$$H^i(\mathcal{A}(n)_{pq};\mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & \text{for } i = 0, 1, \\ \mathbf{Q}^{\oplus(s_j-1)}, & \text{for } i = 2j, \text{ where } 1 \leqslant j < n, \\ \mathbf{Q}^{\oplus s_j}, & \text{for } i = 2j, \text{ where } j = n, \\ \mathbf{Q}^{\oplus(s_j+1)}, & \text{for } i = 2j, \text{ where } n < j \leqslant 2n, \\ 0, & \text{otherwise,} \end{cases}$$

where s_j, $j \in \mathbf{Z}$ are the coefficients of the polynomial

$$\sum_{j=-\infty}^{\infty} s_j q^j = \binom{n+1}{1}_q^2 := \frac{(1-q^{n+1})^2}{(1-q)^2} \in \mathbf{Z}[q]. \tag{6.4}$$

We also have partial results for the integral cohomology of $\mathcal{A}(n)_{pq}$.

Theorem 6.4.2. *The integral cohomology of $\mathcal{A}(n)_{pq}$ in dimensions* 0, 1, 2, *and* 3 *is given by*

$$H^i(\mathcal{A}(n)_{pq}) \cong \begin{cases} \mathbf{Z}, & \text{for } i = 0, \\ \mathbf{Z}, & \text{for } i = 1, \\ \mathbf{Z}, & \text{for } i = 2, \\ \mathbf{Z}_{(p-1,q-1)}, & \text{for } i = 3, \end{cases}$$

where $(p-1, q-1)$ denotes the greatest common divisor of $p-1$ and $q-1$.

Notation 6.4.3. Within this section, we will let the symbols d and e denote p and q in some order. That is to say, $\{d, e\} = \{p, q\}$.

Constructing the Mayer-Vietoris sequence. In the case of the degree-pq component of the $\mathcal{A}$-space, we have described a model for $\mathcal{A}(n)_{pq}$ in Section 3.4.6 as the quotient

$$\mathcal{A}(n)_{pq} \cong \frac{\widetilde{\mathcal{A}}(n)_{p,q} \amalg \widetilde{\mathcal{A}}(n)_{q,p}}{\sim_{pq}},$$

where the equivalence relation $\sim_{pq}$ is generated by the relations

$$\widetilde{\mathcal{A}}(n)_{d,e} \ni [v, w] \sim_{pq} [w, v] \in \widetilde{\mathcal{A}}(n)_{e,d} \quad \text{if} \quad v \perp w \text{ or } v \parallel w.$$

The space $\widetilde{\mathcal{A}}(n)_{d,e}$ is a smooth manifold of (real) dimension $2n$ (c.f. the p^2 case in Section 6.2), and the subspaces $\Delta_{d,e}, \Delta^-_{d,e} \subseteq \widetilde{\mathcal{A}}(n)_{d,e}$ are smooth submanifolds of codimension n and 1 respectively. Hence, there exist tubular neighbourhoods $\widetilde{N}_{d,e}$ and $\widetilde{N}^-_{d,e}$ of $\Delta_{d,e}$ and $\Delta^-_{d,e}$ respectively. Now:

- Denote the image of $\widetilde{\mathcal{A}}(n)_{d,e} \amalg (\widetilde{N}_{e,d} \sqcup \widetilde{N}^-_{e,d})$ in the quotient $\mathcal{A}(n)_{pq}$ by $N_{d,e}$. Then $N_{d,e}$ a regular neighbourhood of the image $\mathcal{A}(n)_{d,e}$ of $\widetilde{\mathcal{A}}(n)_{d,e}$ in $\mathcal{A}(n)_{pq}$.
- $N_{p,q}$ and $N_{q,p}$ is a cover of $\mathcal{A}(n)_{pq}$, with $\Delta \sqcup \Delta^- \subseteq \mathcal{A}(n)_{pq}$ a deformation retract of their intersection $N_{p,q} \cap N_{q,p}$.

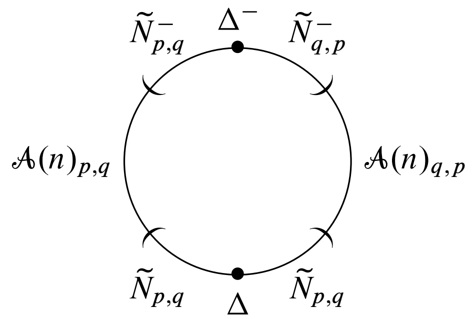

Figure 6.2: Schematic diagram of $\mathcal{A}(n)_{pq}$.

The resulting Mayer-Vietoris sequence is the following:

$$\begin{aligned}
0 \longrightarrow H^0(\mathcal{A}(n)_{pq}) &\longrightarrow H^0(\mathcal{A}(n)_{p,q}) \oplus H^0(\mathcal{A}(n)_{q,p}) \xrightarrow{\Psi} H^0(\Delta) \oplus H^0(\Delta^-) \\
\longrightarrow H^1(\mathcal{A}(n)_{pq}) &\longrightarrow H^1(\mathcal{A}(n)_{p,q}) \oplus H^1(\mathcal{A}(n)_{q,p}) \xrightarrow{\Psi} H^1(\Delta) \oplus H^1(\Delta^-) \\
\longrightarrow H^2(\mathcal{A}(n)_{pq}) &\longrightarrow H^2(\mathcal{A}(n)_{p,q}) \oplus H^2(\mathcal{A}(n)_{q,p}) \xrightarrow{\Psi} H^2(\Delta) \oplus H^2(\Delta^-) \\
\longrightarrow \cdots .
\end{aligned} \tag{6.5}$$

We denote the map $H^i(\mathcal{A}(n)_{p,q}) \oplus H^i(\mathcal{A}(n)_{q,p}) \to H^i(\Delta) \oplus H^i(\Delta^-)$ by Ψ.

Calculating the Mayer-Vietoris sequence

A point in $\mathcal{A}(n)_{d,e}$ is an equivalence class $[v, w]$ representing normal form factorisations of a degree-pq split polynomial with composition ordering $(v, d) \circ (w, e)$. Recalling that $\widetilde{\mathcal{A}}(n)_{d,e} = \mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{e-1}$, we consider the projection $\mathbf{C}P^n \times L^{2n+1}_{e-1} \to L^{2n+1}_{e-1}$. Composing with $L^{2n+1}_{e-1} \to \mathbf{C}P^n$, this descends to a well-defined map

$$\pi : \mathbf{C}P^n \underset{S^1}{\widetilde{\times}} L^{2n+1}_{e-1} \longrightarrow \mathbf{C}P^n$$

on the quotient. (Note that the projection onto first factor $\mathbf{C}P^n$ does *not* descend to the quotient.) Through the homeomorphism $\mathcal{A}(n)_{d,e} \cong \widetilde{\mathcal{A}}(n)_{d,e}$, we replace the domain $\mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{e-1}$ of π with $\mathcal{A}(n)_{d,e}$; we continue to call this map π. Now, π is precisely the map $[v, w] \mapsto [w]$, which extracts from a normal form factorisation the second map (w, e). Therefore, we can identify the codomain $\mathbf{C}P^n$ with the degree-e $\mathcal{A}$-space $\mathcal{A}(n)_e$.

Lemma 6.4.4. *The map π is a $\mathbf{C}P^n$-bundle*

$$\mathbf{C}P^n \longrightarrow \mathcal{A}(n)_{d,e} \xrightarrow{\pi} \mathcal{A}(n)_e.$$

Proof. Letting e_0 denote the first basis vector of $\mathbf{C}^{n+1}$, the fibre of π is given by

$$\pi^{-1}([e_0]) = \{\, [v, \mu e_0] \mid v \in S^{2n+1},\ \mu \in S^1 \,\}.$$

Now, noting that each equivalence class $[v, \mu e_0] \in \mathbf{C}P^n \widetilde{\times}_{S^1} L^{2n+1}_{e-1}$ consists of the elements $([A_{e_0}^{\lambda^{1-e}} v], [\lambda\mu e_0]) \in \mathbf{C}P^n \times L^{2n+1}_{e-1}$, where $\lambda \in S^1$, there is a well-defined map

$$\begin{array}{ccc} \pi^{-1}([e_0]) & \longrightarrow & \mathbf{C}P^n \\ [v, \mu e_0] & \longmapsto & [A_{e_0}^{\mu^{e-1}} v]. \end{array}$$

We check that this is bijective. Each $v \in S^{2n+1}$ admits an orthogonal decomposition

$$v = v_{\parallel} + v_{\perp} \quad \text{where} \quad v_{\parallel} \in \mathbf{C}e_0,\ v_{\perp} \in (\mathbf{C}e_0)^{\perp}.$$

So if $[A_{e_0}^{\mu_1^{e-1}} v_1] = [A_{e_0}^{\mu_2^{e-1}} v_2]$, then we must have $\mu_1^{e-1} v_{1\parallel} = \lambda \mu_2^{e-1} v_{2\parallel}$ and $v_{1\perp} = \lambda v_{2\perp}$ for some $\lambda \in S^1$. Therefore,

$$[v_{1\parallel} + v_{1\perp}, \mu_1 e_0] = [\lambda(\mu_2/\mu_1)^{e-1} v_{2\parallel} + \lambda v_{2\perp}, \mu_1 e_0] = [v_{2\parallel} + v_{2\perp}, \mu_2 e_0].$$

We conclude that $\pi^{-1}([e_0]) \to \mathbf{C}P^n$ is a homeomorphism, giving us the required bundle. $\square$

Lemma 6.4.5. *The cohomology groups of $\mathcal{A}(n)_{d,e}$ are*

$$H^i(\mathcal{A}(n)_{d,e}) \cong \begin{cases} \mathbf{Z}^{\oplus s_j}, & \text{for } i = 2j, \text{ where } 0 \leqslant j \leqslant 2n, \\ 0, & \text{otherwise}, \end{cases}$$

where s_j are the integer defined in equation (6.4).

Proof. By Lemma 6.4.4, we have a $\mathbf{C}P^n$-bundle $\mathbf{C}P^n \to \mathcal{A}(n)_{d,e} \to \mathbf{C}P^n$ via the homeomorphism $\mathcal{A}(n)_e \cong \mathbf{C}P^n$. We apply the Serre spectral sequence to this fibre bundle. Because the base $\mathbf{C}P^n$ is simply connected, the spectral sequence has untwisted coefficients. The E_2 page is the following:

$$\begin{array}{c|cccccc}
t & & & & & & \\
2n & \mathbf{Z}y^n & \mathbf{Z}xy^n & \mathbf{Z}x^2y^n & \cdots & \mathbf{Z}x^{n-1}y^n & \mathbf{Z}x^ny^n \\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\
2 & \mathbf{Z}y & \mathbf{Z}xy & \mathbf{Z}x^2y & \cdots & \mathbf{Z}x^{n-1}y & \mathbf{Z}x^ny \\
0 & \mathbf{Z} & \mathbf{Z}x & \mathbf{Z}x^2 & \cdots & \mathbf{Z}x^{n-1} & \mathbf{Z}x^n \\
\hline
 & 0 & 2 & 4 & \cdots & 2n-2 & 2n \quad s
\end{array}$$

The symbols x and y denote the generators of the groups $E_2^{2,0}$ and $E_2^{0,2}$ respectively. In particular:

- x is image of the 1st Chern class of the base $c_1(\gamma_n) \in H^2(\mathbf{C}P^n)$ in $H^2(\mathcal{A}(n)_{d,e})$ under the boundary homomoprhism $H^2(\mathbf{C}P^n) \to H^2(\mathcal{A}(n)_{d,e})$.

- y is a preimage of the 1st Chern class of the fibre $c_1(\gamma_n) \in H^2(\mathbf{C}P^n)$ in $H^2(\mathcal{A}(n)_{d,e})$ under the boundary homomorphism $H^2(\mathcal{A}(n)_{d,e}) \to H^2(\mathbf{C}P^n)$.

All non-zero groups are concentrated in the even dimensions, and therefore there are no non-zero differentials. Because $\mathbf{Z}$ is free, there are no extension problems. Therefore, the spectral sequence collapses immediately on the E_2, with ring structure that of the truncated polynomial ring in two variables $\mathbf{Z}[x, y]/(x^{n+1}, y^{n+1})$; this is the associated graded of $H^*(\mathcal{A}(n)_{d,e})$. We conclude that $H^*(\mathcal{A}(n)_{d,e})$ is a quotient of the polynomial ring $\mathbf{Z}[x, y]$, but we are unable to determine the relations without further investigation. However, as abelian groups we have

$$H^i(\mathcal{A}(n)_{d,e}) \cong \begin{cases} \mathbf{Z}^{\oplus s_j}, & \text{for } i = 2j, \text{ where } 0 \leqslant j \leqslant 2n, \\ 0, & \text{otherwise,} \end{cases}$$

as desired. ☐

Lemma 6.4.6. *The cohomology groups of Δ^- are given by*

$$H^i(\Delta^-) \cong \begin{cases} \mathbf{Z}^{\oplus r_j}, & \textit{for } i = 2j, \textit{ where } j = 0, \\ \mathbf{Z}^{\oplus(r_j + r_{j-1})}, & \textit{for } i = 2j, \textit{ where } j = 1, 2, \ldots, 2n-2, \\ \mathbf{Z}^{\oplus r_{j-1}}, & \textit{for } i = 2j, \textit{ where } j = 2n-1, \\ 0, & \textit{otherwise,} \end{cases}$$

where r_j are integers defined in equation (6.1).

Proof. Recall that Δ^- is homeomorphic to the anti-diagonal $\Delta^-_{\mathbf{C}P^n} \subseteq \mathbf{C}P^n \times \mathbf{C}P^n$ of the product. There is a $\mathbf{C}P^{n-1}$-bundle $\mathbf{C}P^{n-1} \to \Delta^-_{\mathbf{C}P^n} \to \mathbf{C}P^n$, where the map $\Delta^-_{\mathbf{C}P^n} \to \mathbf{C}P^n$ is the projection onto one of the factors. (Note that the factor we project onto doesn't matter here, but taking $\Delta^- \subseteq \mathcal{A}(n)_{d,e}$ as a subspace, we have a canonical choice by restricting $\pi : \mathcal{A}(n)_{d,e} \to \mathcal{A}(n)_e \cong \mathbf{C}P^n$ to Δ^-.) Via the homeomorphism $\Delta^- \cong \Delta^-_{\mathbf{C}P^n}$, we can replace $\Delta^-_{\mathbf{C}P^n}$ with Δ^- to get the bundle $\mathbf{C}P^{n-1} \to \Delta^- \to \mathbf{C}P^n$ to which we apply the Serre spectral sequence. The base $\mathbf{C}P^n$ is simply connected, and therefore the spectral sequence has untwisted coefficients. The E_2 page is given by:

$$
\begin{array}{c|cccccc}
t & & & & & & \\
2n-2 & \mathbf{Z}b^{n-1} & \mathbf{Z}ab^{n-1} & \mathbf{Z}a^2b^{n-1} & \cdots & \mathbf{Z}a^{n-1}b^{n-1} & \mathbf{Z}a^nb^{n-1} \\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\
2 & \mathbf{Z}b & \mathbf{Z}ab & \mathbf{Z}a^2b & \cdots & \mathbf{Z}a^{n-1}b & \mathbf{Z}a^nb \\
0 & \mathbf{Z} & \mathbf{Z}a & \mathbf{Z}a^2 & \cdots & \mathbf{Z}a^{n-1} & \mathbf{Z}a^n \\
\hline
& 0 & 2 & 4 & \cdots & 2n-2 & 2n \quad s
\end{array}
$$

The symbols a and b denote the generators of the groups $E_2^{2,0}$ and $E_2^{0,2}$ respectively. Like in case for $\mathcal{A}(n)_{d,e}$ (see the proof of Lemma 6.4.5):

- a is image of the 1st Chern class of the base $c_1(\gamma_n) \in H^2(\mathbf{C}P^n)$ in $H^2(\Delta^-)$ under the boundary homomoprhism $H^2(\mathbf{C}P^n) \to H^2(\Delta^-)$.
- b is a preimage of the 1st Chern class of the fibre $c_1(\gamma_{n-1}) \in H^2(\mathbf{C}P^{n-1})$ in $H^2(\Delta^-)$ under the boundary homomorphism $H^2(\Delta^-) \to H^2(\mathbf{C}P^{n-1})$.

All non-zero groups are concentrated in the even dimensions, and therefore there are no non-zero differentials. Because $\mathbf{Z}$ is free, there are no extension problems. Therefore, the spectral sequence collapses immediately on the E_2, with ring structure that of the truncated polynomial ring in two variables $\mathbf{Z}[a,b]/(a^{n+1}, b^n)$; this is the associated graded of $H^*(\Delta^-)$. We conclude that $H^*(\Delta^-)$ is a quotient of the polynomial ring $\mathbf{Z}[a,b]$, but we are unable to determine the relations without further investigation. As abelian groups, we have

$$
H^i(\Delta^-) \cong \begin{cases}
\mathbf{Z}^{\oplus r_j}, & \text{for } i = 2j, \text{ where } j = 0, \\
\mathbf{Z}^{\oplus(r_j + r_{j-1})}, & \text{for } i = 2j, \text{ where } j = 1, 2, \ldots, 2n-2, \\
\mathbf{Z}^{\oplus r_{j-1}}, & \text{for } i = 2j, \text{ where } j = 2n-1, \\
0, & \text{otherwise,}
\end{cases}
$$

as desired. $\square$

Remark 6.4.7. In fact, we have a description of the two generators of $H^2(\Delta^-)$ by Theorem 5.2.3. Thinking of the bundle $\mathbf{C}P^{n-1} \to \Delta^- \to \mathbf{C}P^n$ as the restriction of $\mathbf{C}P^n \to \mathcal{A}(n)_{d,e} \to \mathcal{A}(n)_e$ to Δ^-, we see that the projection is onto the degree-e component of the $\mathcal{A}$-space. By Theorem 5.2.8, the bundle over this space has the eth tensor power of γ_n, and therefore the pullback of γ_n along the projection $\Delta^- \to \mathbf{C}P^n$ must be $\kappa_{e,n}$. In the notation of the proof above, its 1st Chern class $c_1(\kappa_{e,n})$

corresponds to a. The fibre $\mathbf{C}P^{n-1}$ consists of planes orthogonal to the chosen basepoint, and hence we obtain γ_{n-1} by pulling back $\kappa_{d,e}$ along the inclusion $\mathbf{C}P^{n-1} \to \Delta^-$. Its 1st Chern class $c_1(\kappa_{d,n})$ corresponds to b.

Calculating Ψ. The map $\Psi : H^i(\mathcal{A}(n)_{p,q}) \oplus H^i(\mathcal{A}(n)_{q,p}) \to H^i(\Delta) \oplus H^i(\Delta^-)$ has a description as the block matrix

$$\begin{array}{ccc} & \begin{array}{cc} H^i(\mathcal{A}(n)_{p,q}) & H^i(\mathcal{A}(n)_{q,p}) \end{array} & \\ \Psi = & \begin{pmatrix} j^*_{p,q} & -j^*_{q,p} \\ k^*_{p,q} & -k^*_{q,p} \end{pmatrix} & \begin{array}{c} H^i(\Delta) \\ H^i(\Delta^-) \end{array} \end{array}$$

where $j_{d,e} : \Delta \hookrightarrow \mathcal{A}(n)_{d,e}$ and $k_{d,e} : \Delta^- \hookrightarrow \mathcal{A}(n)_{d,e}$ are the inclusion maps.

Recall that the proof of Lemma 6.4.5 finds generators $x, y \in H^2(\widetilde{\mathcal{A}}(n)_{d,e})$ of the cohomology ring $H^*(\widetilde{\mathcal{A}}(n)_{d,e})$, and Remark 6.4.7 tells us that $c_1(\kappa_{p,n}), c_1(\kappa_{q,n}) \in H^2(\Delta^-)$ are the generators of the cohomology ring $H^*(\Delta^-)$. The cohomology ring $H^*(\Delta)$ is generated by $c_1(\gamma_n) \in H^2(\Delta)$. To disambiguate, we denote the generators of $H^2(\mathcal{A}(n)_{p,q})$ by x, y, and the generators of $H^2(\mathcal{A}(n)_{q,p})$ by x', y' when appropriate. In this notation, we state the following lemma.

Lemma 6.4.8. *In matrix form, the map $\Psi : H^{2j}(\mathcal{A}(n)_{p,q}) \oplus H^{2j}(\mathcal{A}(n)_{q,p}) \to H^{2j}(\Delta) \oplus H^{2j}(\Delta^-)$ for $j \geqslant 0$ is given by*

$$\begin{pmatrix} j^*_{p,q} & -j^*_{q,p} \\ k^*_{p,q} & -k^*_{q,p} \end{pmatrix} \begin{pmatrix} \sum_{s=0}^{j} a_s x^s y^{j-s} \\ \sum_{s=0}^{j} b_s x'^s y'^{j-s} \end{pmatrix} = \begin{pmatrix} \sum_{s=0}^{j} (a_s q^{j-s} - b_s p^{j-s})\, c_1(\gamma_n)^j \\ \sum_{s=0}^{j} (a_s - b_{j-s})\, c_1(\kappa_{q,n})^s c_1(\kappa_{p,n})^{j-s} \end{pmatrix},$$

where $a_s, b_s \in \mathbf{Z}$.

Proof. This proof will involve a few lengthy calculations analysing the inclusions $j_{d,e}$ and $k_{d,e}$.

A description of $k^*_{d,e}$. We begin by considering the inclusion $k_{d,e} : \Delta^- \hookrightarrow \mathcal{A}(n)_{d,e}$. Recall that $\mathcal{A}(n)_{d,e}$ is a $\mathbf{C}P^n$-bundle $\mathbf{C}P^n \to \mathcal{A}(n)_{d,e} \to \mathbf{C}P^n$ by Lemma 6.4.4, where the map $\mathcal{A}(n)_{d,e} \to \mathbf{C}P^n$ is quotient of the projection map $\mathbf{C}P^n \times L^{2n+1}_{e-1} \to L^{2n+1}_{e-1}$. Restricted to the subspace Δ^-, we instead have the $\mathbf{C}P^{n-1}$-bundle $\mathbf{C}P^{n-1} \to \Delta^- \to \mathbf{C}P^n$ (see proof of Lemma 6.4.6). These fibre bundles fit into the commutative diagram

$$\begin{tikzcd} \mathbf{C}P^{n-1} \arrow[r, hook] \arrow[d, hook] & \mathbf{C}P^n \arrow[d, hook] \\ \Delta^- \arrow[r, hook, "k_{d,e}"] \arrow[d] & \mathcal{A}(n)_{d,e} \arrow[d] \\ \mathbf{C}P^n \arrow[r, equal] & \mathbf{C}P^n, \end{tikzcd}$$

where the map on the fibres $\mathbf{C}P^{n-1} \hookrightarrow \mathbf{C}P^n$ is the inclusion induced by $\mathbf{C}^n \hookrightarrow \mathbf{C}^{n+1}$. This induces isomorphisms

$$H^i(\mathbf{C}P^n) \cong H^i(\mathbf{C}P^{n-1})$$

for all $i \leqslant 2n-2$, while the bottom horizontal map induces isomorphisms on all cohomology groups. Denoting the E_2 page of the Serre spectral sequence for $\mathbf{C}P^n \to \mathcal{A}(n)_{d,e} \to \mathbf{C}P^n$ (see proof of Lemma 6.4.5) by E_2, and for $\mathbf{C}P^{n-1} \to \Delta^- \to \mathbf{C}P^n$ (see proof of Lemma 6.4.6) by $'E_2$, we have by naturality of the Serre spectral sequence induced natural isomorphisms

$$'E_2^{s,t} \cong E_2^{s,t}$$

for all $p+q \leqslant 2n-2$. Both spectral sequences degenerate on the E_2 page because all non-zero cohomology groups of $\mathbf{C}P^n$ and $\mathbf{C}P^{n-1}$ are concentrated in the even dimensions. So $k_{d,e}$ induces isomorphisms

$$H^i(\mathcal{A}(n)_{d,e}) \cong H^i(\Delta^-)$$

for all $i \leqslant 2n-2$. In the notation of the proofs for Lemmas 6.4.5 and 6.4.6, we remark that $k_{d,e}^*$ maps $x \in H^2(\widetilde{\mathcal{A}}(n)_{d,e})$ to $a \in H^2(\Delta^-)$ and $y \in H^2(\widetilde{\mathcal{A}}(n)_{d,e})$ to $b \in H^2(\Delta^-)$.

A description of $j_{d,e}^*$. For the other inclusion $j_{d,e} : \Delta \hookrightarrow \mathcal{A}(n)_{d,e}$, we begin by determining $j_{d,e}^*$ on H^2. From Section 5.2.2, we know that there is a principal $U(n+1)$-bundle $\rho : SP(n) \to \mathcal{A}(n)$, and therefore it has a classifying map $\phi_\rho : \mathcal{A}(n) \to BU(n+1)$. Restricting this bundle to the subspace $\mathcal{A}(n)_{d,e}$, and then further to the subspaces $\Delta, \Delta^- \subseteq \mathcal{A}(n)_{d,e}$, we find the commutative diagram

$$\begin{tikzcd}
\Delta^- \arrow[r, hook, "k_{d,e}"] \arrow[d, "\phi_\rho|_{\Delta^-}"'] & \mathcal{A}(n)_{d,e} \arrow[dl, "\phi_\rho|_{\mathcal{A}(n)_{d,e}}"] \\
BU(n+1) & \Delta. \arrow[l, "\phi_\rho|_\Delta"] \arrow[u, hook, "j_{d,e}"']
\end{tikzcd}$$

This yields the following commutative diagram on H^2:

$$\begin{tikzcd}
H^2(\Delta^-) & H^2(\mathcal{A}(n)_{d,e}) \arrow[l, "k_{d,e}^*"', "\cong"] \arrow[d, "j_{d,e}^*"] \\
H^2(BU(n+1)) \arrow[u, "(\phi_\rho|_{\Delta^-})^*"] \arrow[ur, "(\phi_\rho|_{\mathcal{A}(n)_{d,e}})^*"] \arrow[r, "(\phi_\rho|_\Delta)^*"'] & H^2(\Delta).
\end{tikzcd} \tag{6.6}$$

Recall that $H^2(BU(n+1)) \cong \mathbf{Z}$ is generated by the 1st Chern class $c_1(VU(n+1))$ [MS74, Theorem 14.5], and $H^2(\Delta) \cong \mathbf{Z}$ is generated by the 1st Chern class $c_1(\gamma_n)$ (c.f. Lemma 6.2.3). We compute the bottom horizontal map

$$H^2(BU(n+1)) \xrightarrow{(\phi_\rho|_\Delta)^*} H^2(\Delta(n))$$

in two different ways around the diagram (6.6) by tracking how it acts on the Chern classes:

1. By Theorem 5.2.8, the associated vector bundle $V\rho : V(SP(n)) \to \mathcal{A}(n)$ restricts to $\gamma_n^{\otimes d} \oplus \gamma_n^\perp \to \mathcal{A}(n)_d^{\mathrm{at}}$ over the degree-d atomic $\mathcal{A}$-space. However, we now notice that Δ is precisely the atomic $\mathcal{A}$-space of degree pq, for an equivalence class $[\lambda v, \mu v] \in \Delta$ consists of those

split polynomials which can be expressed as $A \circ (\lambda v, p) \circ (\mu v, q) = AA^{\lambda^{1-p}\mu^{p(1-q)}} \circ (v, pq)$ or $A \circ (\lambda v, q) \circ (\mu v, p) = AA^{\lambda^{1-q}\mu^{q(1-p)}} \circ (v, pq)$ for some $A \in U(n+1)$. Therefore the restriction $\phi_\rho|_\Delta : \Delta \to BU(n+1)$ is the classifying map for the vector bundle $\gamma_n^{\otimes pq} \oplus \gamma_n^\perp \to \Delta$. By naturality, the pullback of the 1st Chern class of $VU(n+1)$ is thus

$$(\phi_\rho|_\Delta)^* c_1(VU(n+1)) = c_1(\gamma_n^{\otimes pq}) + c_1(\gamma_n^\perp) = (pq-1)\, c_1(\gamma_n) \in H^2(\Delta).$$

2. The map $k_{d,e}^*$ gives isomorphisms

$$H^2(\Delta^-) \cong H^2(\mathcal{A}(n)_{d,e}) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

The proof of Lemma 6.4.5 gives explicit generators $x, y \in H^2(\mathcal{A}(n)_{d,e})$: x is the 1st Chern class of the pullback of γ_n along the projection $\mathcal{A}(n)_{d,e} \to \mathbf{C}P^n$ onto the second factor, and y is a preimage of $c_1(\gamma_n)$ under $H^2(\mathcal{A}(n)_{d,e}) \to H^2(\mathbf{C}P^n)$. But now, the diagonal Δ receives a homeomorphism from $\mathbf{C}P^n$ such that the composition

$$\mathbf{C}P^n \xrightarrow{\cong} \Delta \xhookrightarrow{j_{d,e}} \mathcal{A}(n)_{d,e} \twoheadrightarrow \mathbf{C}P^n$$

is the identity. We deduce that the image of x under $j_{d,e}^* : H^2(\mathcal{A}(n)_{d,e}) \to H^2(\Delta)$ must be $c_1(\gamma_n)$.

It remains to calculate the image of y in $H^2(\Delta)$ under $j_{d,e}^*$. By Theorem 5.2.3, $V\rho : V(SP(n)) \to \mathcal{A}(n)$ restricts to $\kappa_{d,n}^{\otimes d} \oplus \kappa_{e,n}^{\otimes e} \oplus (\kappa_{d,n} \oplus \kappa_{e,n})^\perp \to \Delta^-$ over the anti-diagonal, and $\phi_\rho|_{\Delta^-}$ is its classifying map. By Remark 6.4.7, $c_1(\kappa_{d,e})$ and $c_1(\kappa_{e,n})$ are the images of the two generators y and x respectively under $k_{d,e}^*$. Let the image of y under $j_{d,e}^*$ be $m\, c_1(\gamma_n) \in H^2(\Delta)$ for some $m \in \mathbf{Z}$. First, computing the composition

$$H^2(BU(n+1)) \xrightarrow{(\phi_\rho|_{\Delta^-})^*} H^2(\Delta^-) \xrightarrow[\cong]{k_{d,e}^{*-1}} H^2(\mathcal{A}(n)_{d,e})$$

yields

$$\begin{aligned} k_{d,e}^{*-1}(\phi_\rho|_{\mathcal{A}(n)_{d,e}})^* c_1(VU(n+1)) & \\ &= k_{d,e}^{*-1}(c_1(\kappa_{d,n}^{\otimes d}) + c_1(\kappa_{e,n}^{\otimes e}) - c_1(\kappa_{d,n}) - c_1(\kappa_{e,n})) \\ &= (d-1)\, k_{d,e}^{*-1} c_1(\kappa_{d,n}) + (e-1)\, k_{d,e}^{*-1} c_1(\kappa_{e,n}) \\ &= (d-1)y + (e-1)x. \end{aligned}$$

Further applying $j_{d,e}^*$, we find that

$$j_{d,e}^*((d-1)y + (e-1)x) = (m(d-1) + (e-1))\, c_1(\gamma_n).$$

By commutativity of (6.6), we must have

$$m(d-1) + (e-1) = pq - 1 = de - 1.$$

Therefore, $m = e$. We conclude that $j_{d,e}^* : H^2(\mathcal{A}(n)_{d,e}) \to H^2(\Delta)$ is given by

$$j_{d,e}^*(ax + by) = (a + eb)\, c_1(\gamma_n)$$

for $a, b \in \mathbf{Z}$.

Then, the map $\Psi : H^2(\mathcal{A}(n)_{p,q}) \oplus H^2(\mathcal{A}(n)_{q,p}) \to H^2(\Delta) \oplus H^2(\Delta^-)$ is given by

$$\begin{pmatrix} j_{p,q}^* & -j_{q,p}^* \\ k_{p,q}^* & -k_{q,p}^* \end{pmatrix} \begin{pmatrix} ax + by \\ cx' + dy' \end{pmatrix} = \begin{pmatrix} (a - c + qb - pd)\, c_1(\gamma_n) \\ (b - c)\, c_1(\kappa_{p,n}) + (a - d)\, c_1(\kappa_{q,n}) \end{pmatrix}$$

for $a, b, c, d \in \mathbf{Z}$.

To determine $j_{d,e}^*$ on the higher cohomology groups, we recall that a map of spaces $j_{d,e} : \Delta \hookrightarrow \mathcal{A}(n)_{d,e}$ induces a map of rings $j_{d,e}^* : H^*(\mathcal{A}(n)_{d,e}) \to H^*(\Delta)$. Therefore, we also immediately conclude that

$$j_{d,e}^*(x^s y^t) = e^t\, c_1(\gamma_n)^{s+t}$$

for $0 \leqslant s, t \leqslant n$. The map $\Psi : H^{2j}(\mathcal{A}(n)_{p,q}) \oplus H^{2j}(\mathcal{A}(n)_{q,p}) \to H^{2j}(\Delta) \oplus H^{2j}(\Delta^-)$ has the form

$$\begin{pmatrix} j_{p,q}^* & -j_{q,p}^* \\ k_{p,q}^* & -k_{q,p}^* \end{pmatrix} \begin{pmatrix} \sum\limits_{s=0}^{j} a_s x^s y^{j-s} \\ \sum\limits_{s=0}^{j} b_s x'^s y'^{j-s} \end{pmatrix} = \begin{pmatrix} \sum\limits_{s=0}^{j} (a_s q^{j-s} - b_s p^{j-s})\, c_1(\gamma_n)^j \\ \sum\limits_{s=0}^{j} (a_s - b_{j-s})\, c_1(\kappa_{q,n})^s c_1(\kappa_{p,n})^{j-s} \end{pmatrix},$$

where $a_s, b_s \in \mathbf{Z}$. ☐

Proof of Theorems 6.4.1 and 6.4.2. We return the Mayer-Vietoris sequence (6.5). Because the cohomology groups of $\mathcal{A}(n)_{d,e}$, Δ, and Δ^- are all concentrated in the even dimensions, the Mayer-Vietoris sequence becomes disjoint exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[dll, out=-10, in=170] & \\
H^{2j}(\mathcal{A}(n)_{pq}) \arrow[r] & H^{2j}(\mathcal{A}(n)_{p,q}) \oplus H^{2j}(\mathcal{A}(n)_{q,p}) \arrow[r, "\Psi^{2j}"] & H^{2j}(\Delta) \oplus H^{2j}(\Delta^-) \arrow[dll, out=-10, in=170] \\
H^{2j+1}(\mathcal{A}(n)_{pq}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

for all j. To disambiguate, we decorate the map $\Psi : H^{2j}(\mathcal{A}(n)_{p,q}) \oplus H^{2j}(\mathcal{A}(n)_{q,p}) \to H^{2j}(\Delta) \oplus H^{2j}(\Delta^-)$ in dimension $2j$ with an index of $2j$. By exactness, we have isomorphisms $H^{2j}(\mathcal{A}(n)_{pq}) \cong \ker \Psi^{2j}$ and $H^{2j+1}(\mathcal{A}(n)_{pq}) \cong \operatorname{coker} \Psi^{2j}$.

- For $j = 0$, Ψ^0 is the diagonal map

$$\begin{pmatrix} j_{p,q}^* & -j_{q,p}^* \\ k_{p,q}^* & -k_{q,p}^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - b \\ a - b \end{pmatrix}.$$

  So $\ker \Psi^0 \cong \mathbf{Z}$ and $\operatorname{coker} \Psi^0 \cong \mathbf{Z}$.

- For $j = 1$, Ψ^2 can be written as the integer matrix

$$\Psi^2 = \begin{array}{cc} & \begin{array}{cccc} x & y & x' & y' \end{array} \\ & \begin{pmatrix} 1 & q & -1 & -p \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{array}{l} c_1(\gamma_n) \\ c_1(\kappa_{p,n}) \\ c_1(\kappa_{q,n}) \end{array} \end{array}$$

with respect to the basis x, y, x', y' for $H^2(\mathcal{A}(n)_{p,q}) \oplus H^2(\mathcal{A}(n)_{q,p})$, and the basis $c_1(\gamma_n)$, $c_1(\kappa_{p,n}), c_1(\kappa_{q,n})$ for $H^2(\Delta) \oplus H^2(\Delta^-)$. Row reducing, the Smith normal form of the matrix of Ψ^2 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (p-1, q-1) & 0 \end{pmatrix}.$$

Therefore, $\ker \Psi^2 \cong \mathbf{Z}$ and $\operatorname{coker} \Psi^2 \cong \mathbf{Z}_{(p-1,q-1)}$.

This proves Theorem 6.4.2.

- In general, we make the computation rationally. Tensoring with $\mathbf{Q}$, $\Psi^{2j} \otimes \mathbf{Q}$ is surjective in all dimensions except 0. So for $j > 0$, we have $\operatorname{coker} \Psi^{2j} = 0$, and by rank-nullity,

$$\ker \Psi^{2j} \cong \begin{cases} \mathbf{Q}^{\oplus(s_j - 1)}, & \text{if } 1 \leqslant j < n, \\ \mathbf{Q}^{\oplus s_j}, & \text{if } j = n, \\ \mathbf{Q}^{\oplus(s_j + 1)}, & \text{if } n < j \leqslant 2n, \\ 0, & \text{otherwise,} \end{cases}$$

where we recall that $s_j = \operatorname{rank} H^{2j}(\mathcal{A}(n)_{p,q}) = \operatorname{rank} H^{2j}(\mathcal{A}(n)_{q,p})$ by Lemma 6.4.5.

Hence,

$$H^i(\mathcal{A}(n)_{pq}; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & \text{for } i = 0, 1, \\ \mathbf{Q}^{\oplus(s_j - 1)}, & \text{for } i = 2j, \text{ where } 1 \leqslant j < n, \\ \mathbf{Q}^{\oplus s_j}, & \text{for } i = 2j, \text{ where } j = n, \\ \mathbf{Q}^{\oplus(s_j + 1)}, & \text{for } i = 2j, \text{ where } n < j \leqslant 2n, \\ 0, & \text{otherwise,} \end{cases}$$

as desired. □

6.5 The stable cohomology of $\mathcal{A}_{pq}$

In this section, we prove that the cohomology groups of $\mathcal{A}(n)_{pq}$ stabilise, analogous to the results in Section 6.3.

Theorem 6.5.1 (Stable cohomology of $\mathcal{A}_{pq}$). *The inclusion $\mathcal{A}(n)_{p^2} \hookrightarrow \mathcal{A}(n+1)_{p^2}$ induces isomorphisms*

$$H^i(\mathcal{A}(n+1)_{p^2}) \cong H^i(\mathcal{A}(n)_{p^2})$$

for all $i \leqslant 2n - 2$. Furthermore, the rational cohomology groups of $\mathcal{A}_{pq}$ are given by

$$H^i(\mathcal{A}_{pq}; \mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & \textit{for } i = 0, 1, \\ \mathbf{Q}^{\oplus j}, & \textit{for } i = 2j, \textit{ where } j \geqslant 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Notation 6.5.2. For the rest of this section, we denote the diagonal and anti-diagonal of $\mathcal{A}(n)_{pq}$ by $\Delta(n)$ and $\Delta^-(n)$ respectively.

Lemma 6.5.3. *The inclusions* $\Delta(n) \hookrightarrow \Delta(n+1)$ *induce isomorphisms*

$$H^i(\Delta(n+1)) \cong H^i(\Delta(n))$$

for all $i \leqslant 2n$.

Proof. The proof for Lemma 6.3.4 works in this case verbatim. □

Lemma 6.5.4. *The inclusion* $\Delta^-(n) \hookrightarrow \Delta^-(n+1)$ *induces isomorphisms*

$$H^i(\Delta^-(n+1)) \cong H^i(\Delta^-(n))$$

for all $i \leqslant 2n$.

Proof. The proof for Lemma 6.3.5 goes through after replacing every instance of $\mathbf{R}P^2$ with $\mathbf{C}P^1$, and every instance of $\Delta^-/\mathbf{Z}_2$ with Δ^-. □

Lemma 6.5.5. *The inclusion* $\mathcal{A}(n)_{d,e} \hookrightarrow \mathcal{A}(n+1)_{d,e}$ *induces isomorphisms*

$$H^i(\mathcal{A}(n+1))_{d,e} \cong H^i(\mathcal{A}(n))_{d,e}$$

for all $i \leqslant 2n-2$.

Proof. The inclusion $\mathcal{A}(n)_{d,e} \hookrightarrow \mathcal{A}(n+1)_{d,e}$ induces the following inclusion of fibre bundles

$$\begin{tikzcd}
\mathbf{C}P^n \arrow[r, hook] \arrow[d, hook] & \mathbf{C}P^{n+1} \arrow[d, hook] \\
\mathcal{A}(n)_{d,e} \arrow[r, hook] \arrow[d, two heads] & \mathcal{A}(n+1)_{d,e} \arrow[d, two heads] \\
\mathbf{C}P^n \arrow[r, hook] & \mathbf{C}P^{n+1},
\end{tikzcd}$$

where both inclusions $\mathbf{C}P^n \hookrightarrow \mathbf{C}P^{n+1}$ on the fibre and on the base correspond to the inclusion induced by $\mathbf{C}^{n+1} \hookrightarrow \mathbf{C}^{n+2}$. These induce isomorphisms

$$H^i(\mathbf{C}P^{n-1}) \cong H^i(\mathbf{C}P^n)$$

for all $i \leqslant 2n-2$, and therefore by naturality of the Serre spectral sequence (c.f. the proofs in Section 6.3), the inclusion $\mathcal{A}(n)_{d,e} \hookrightarrow \mathcal{A}(n+1)_{d,e}$ induces isomorphisms

$$H^i(\mathcal{A}(n))_{d,e} \cong H^i(\mathcal{A}(n+1))_{d,e}$$

for all $i \leqslant 2n-2$. □

The above lemmas once again provide us with a range in which the cohomology groups of the Mayer-Vietoris sequence (6.5) are stable (c.f. proof of Theorem 6.4.1).

Proof of Theorem 6.5.1. Considering the Mayer-Vietoris sequences for $\mathcal{A}(n)_{pq}$ and $\mathcal{A}(n+1)_{pq}$, we have by naturality a commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{i-1}(\mathcal{A}(n+1)_{p,q}) \oplus H^{i-1}(\mathcal{A}(n+1)_{q,p}) \arrow[r, "\cong"] \arrow[d] & H^{i-1}(\mathcal{A}(n)_{p,q}) \oplus H^{i-1}(\mathcal{A}(n)_{q,p}) \arrow[d] \\
H^{i-1}(\Delta(n+1)) \oplus H^{i-1}(\Delta^{-}(n+1)) \arrow[r, "\cong"] \arrow[d] & H^{i-1}(\Delta(n)) \oplus H^{i-1}(\Delta^{-}(n)) \arrow[d] \\
H^{i}(\mathcal{A}(n+1)_{pq}) \arrow[r] \arrow[d] & H^{i}(\mathcal{A}(n)_{pq}) \arrow[d] \\
H^{i}(\mathcal{A}(n+1)_{p,q}) \oplus H^{i}(\mathcal{A}(n+1)_{q,p}) \arrow[r, "\cong"] \arrow[d] & H^{i}(\mathcal{A}(n)_{p,q}) \oplus H^{i}(\mathcal{A}(n)_{q,p}) \arrow[d] \\
H^{i}(\Delta(n+1)) \oplus H^{i}(\Delta^{-}(n+1)) \arrow[r, "\cong"] \arrow[d] & H^{i}(\Delta(n)) \oplus H^{i}(\Delta^{-}(n)), \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

where 4 out of 5 of the horizontal maps are isomorphisms for all $i \leqslant 2n-2$. Hence, by the 5-lemma, $H^i(\mathcal{A}(n+1)_{pq}) \to H^i(\mathcal{A}(n)_{pq})$ is an isomorphism for all $i \leqslant 2n-2$.

From Theorem 6.4.1, we have for each positive integer n, the rational cohomology of $\mathcal{A}(n)_{pq}$ in the stable range $i \leqslant 2n-2$ is given by

$$H^i(\mathcal{A}(n)_{p^2};\mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & \text{for } i = 0, 1, \\ \mathbf{Q}^{\oplus j}, & \text{for } i = 2j, \text{ where } j \geqslant 1. \end{cases}$$

Hence, by functoriality of cohomology and taking the limit as $n \to \infty$, we find that

$$H^i(\mathcal{A}_{pq};\mathbf{Q}) \cong \begin{cases} \mathbf{Q}, & \text{for } i = 0, 1, \\ \mathbf{Q}^{\oplus j}, & \text{for } i = 2j, \text{ where } j \geqslant 1, \\ 0, & \text{otherwise,} \end{cases}$$

as desired. $\square$

A Some additional proofs

We now provide the omitted proofs relating to the structure of the $\mathcal{A}$-space.

Theorem 3.4.5 (Injectivity of Z for $d = p^2$). *The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_{p^2}}: & \mathcal{A}(n)_{p^2} & \longrightarrow & \{\text{algebraic subsets of } \mathbf{C}^{n+1}\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_{p^2}$ to its set of critical points is injective.

Theorem 3.4.6 (Relations in $\mathcal{A}(n)_{p^2}$). *In the $\mathcal{A}$-space of degree p^2, the following relations are satisfied for all $v, w \in S^{2n+1}$, $\lambda \in S^1$:*

1. $[\lambda v \circ w] = [v \circ w]$ *and* $[v \circ \lambda w] = [A_w^{\lambda^{p-1}} v \circ w]$.
2. $[v \circ w] = [w \circ v]$ *if either* $v \parallel w$ *or* $v \perp w$.

Furthermore, these are the only *relations in* $\mathcal{A}(n)_{p^2}$.

Proof of Theorems 3.4.5 and 3.4.6. To show this, let us first study the structure of $Z[v \circ w]$ for some $[v \circ w] \in \mathcal{A}(n)_{p^2}$. Let V denote the vanishing locus of a polynomial. Then

$$Z[v \circ w] = V(\langle z, w \rangle) \cup V(\langle w(z), v \rangle) = (\mathbf{C}w)^{\perp} \cup w^{-1}((\mathbf{C}v)^{\perp}).$$

We make the following observations:

- The set $w^{-1}((\mathbf{C}v)^{\perp})$ is a hypersurface which is the preimage under w of the the hyperplane $(\mathbf{C}v)^{\perp}$. Explicitly, $w^{-1}((\mathbf{C}v)^{\perp})$ is the set of points satisfying the equation

  $$v_{\perp} z_0 + v_{\parallel} z_n^p = 0,$$

  where $z = z_0 b_0 + \cdots + z_{n-1} b_{n-1} + z_n w$ and $v = v_{\perp} b_0 + v_{\parallel} w$ for some suitable choice of orthonormal basis $b_0, \ldots, b_{n-1}, w$ of $\mathbf{C}^{n+1}$. In particular, $v_{\perp} \neq 0$ as long as $v \nparallel w$, so we can re-express the above equation as

  $$z_0 = -\frac{v_{\parallel}}{v_{\perp}} z_n^p,$$

  which is the graph of the function $(z_1, \ldots, z_n) \mapsto -v_{\parallel} z_n^p / v_{\perp}$ defined on the hyperplane $(\mathbf{C}b_0)^{\perp}$ (spanned by $b_1, \ldots, b_{n-1}, w$) and thinking of the axis spanned by b_0 as the "output" axis. Since $p > 1$, the hypersurface $w^{-1}((\mathbf{C}v)^{\perp})$ will always have the hyperplane $(\mathbf{C}b_0)^{\perp}$ as its tangent hyperplane at the origin, where $b_0 \perp w$.

- When $v \parallel w$, $[v \circ w] \sim [v \circ v]$ and therefore the critical point reduces to a single hyperplane

  $$Z[v \circ w] = Z[(v, p^2)] = V(\langle z, v \rangle) = (\mathbf{C}v)^{\perp}.$$

So from a set of critical points Z of some $[v \circ w] \in \mathcal{A}(n)_{p^2}$, we can decompose it into its irreducible components. There are two cases:

1. If there is only one irreducible component, it is a hyperplane and we can express $Z = (\mathbf{C}v)^\perp$. The only element of $\mathcal{A}(n)_{p^2}$ which is sent to $(\mathbf{C}v)^\perp$ by Z is $[v \cdot w]$ for $v \parallel w$.
2. If there are two irreducible components, one of these components must be a hyperplane $(\mathbf{C}w)^\perp$. The vector $w \in S^{2n+1}$ is unique up to a choice of $\mu \in S^1$, for we have $(\mathbf{C}w)^\perp = (\mathbf{C}\mu w)^\perp$.

The first case leads to the parallel case in relation 2. Let us focus on the second case.

If the second irreducible component is also a hyperplane $(\mathbf{C}v)^\perp$, it is necessary from our observations above that $v \perp w$ (corresponding to the case where $v = v_\perp b_0$ with no w component), in which case the ordering of the hyperplanes does not matter, and we have $Z = Z[v \cdot w] = Z[w \cdot v]$. So assume that the other irreducible component is a hypersurface, which can be described as a graph

$$\Gamma = \{\, c z_n^p b_0 + z_1 b_1 + \cdots + z_{n-1} b_{n-1} + z_n w \mid (z_1, \ldots, z_n) \in \mathbf{C}^n \,\}.$$

The tangent hyperplane to Γ at the origin is $(\mathbf{C}b_0)^\perp$, from which we can reconstruct v using the formula

$$v = v_\perp b_0 + v_\parallel w = -v_\parallel c b_0 + v_\parallel w, \quad \text{where} \quad v_\perp, v_\parallel \in \mathbf{C},\ c = -\frac{v_\perp}{v_\parallel}.$$

We now go through all the choices that we made to check how they affect the resulting pair of vectors v and w that we construct:

- The vector v is unique up to a choice of $\lambda \in S^1$, for we have that

  $$\lambda v = (\lambda v_\perp) b_0 + (\lambda v_\parallel) w = -(\lambda v_\parallel) c b_0 + (\lambda v_\parallel) w$$

  also satisfies $c = -(\lambda v_\perp)/(\lambda v_\parallel)$.

- The choice of b_0 does not matter for if we had chosen $b_0' = \kappa b_0$ instead to represent the hyperplane $(\mathbf{C}b_0)^\perp = (\mathbf{C}\kappa b_0)^\perp$, we can write Γ as

  $$\Gamma = \{\, (\kappa^{-1} c) z_n^p (\kappa b_0) + z_1 b_1 + \cdots + z_{n-1} b_{n-1} + z_n w \mid (z_1, \ldots, z_n) \in \mathbf{C}^n \,\},$$

  and correspondingly the v associated to this representation of Γ is

  $$v = v'_\perp (\kappa b_0) + v'_\parallel w = -v'_\parallel (\kappa^{-1} c)(\kappa b_0) + v'_\parallel w, \ \text{where } v'_\perp, v'_\parallel \in \mathbf{C},\ \kappa^{-1} c = -\frac{v'_\perp}{v'_\parallel},$$

  giving the same v as before.

- The choice of w however does matter. If we had chosen $w' = \mu w$ to represent the hyperplane $(\mathbf{C}w)^\perp = (\mathbf{C}\mu w)^\perp$, we can write Γ as

  $$\Gamma = \{\, (\mu^p c)(z_n \mu^{-1})^p b_0 + z_1 b_1 + \cdots + z_{n-1} b_{n-1} + (z_n \mu^{-1})(\mu w) \mid (z_1, \ldots, z_n) \in \mathbf{C}^n \,\}.$$

  The corresponding v associated to this representation of Γ is

  $$v = v'_\perp b_0 + v'_\parallel (\mu w) = -v'_\parallel (\mu^p c) b_0 + v'_\parallel (\mu w), \quad \text{where} \quad v'_\perp, v'_\parallel \in \mathbf{C},\ \mu^p c = -\frac{v'_\perp}{v'_\parallel}.$$

  So what we find is that

  $$Z = Z[v \circ w] = Z[\lambda v \circ w] = Z[A_w^{\mu^{1-p}} v \circ \mu w], \qquad \lambda, \mu \in S^1.$$

This constructs an inverse map of $Z|_{\mathcal{A}(n)_{p^2}}$ from its image, showing that $Z|_{\mathcal{A}(n)_{p^2}}$ is injective. We have also verified all the relations in $\mathcal{A}(n)_{p^2}$. ☐

Theorem 3.4.11 (Injectivity of Z for $d = pq$). *Let p and q be distinct primes. The map*

$$\begin{array}{rccc} Z|_{\mathcal{A}(n)_{pq}} : & \mathcal{A}(n)_{pq} & \longrightarrow & \{\,\textit{algebraic subsets of}\ \mathbf{C}^{n+1}\,\} \\ & [f] & \longmapsto & Z[f] \end{array}$$

assigning each equivalence class of $\mathcal{A}(n)_{pq}$ to its set of critical points is injective.

Theorem 3.4.12 (Relations in $\mathcal{A}(n)_{pq}$). *Let p and q be distinct primes, and let $\{d, e\} = \{p, q\}$. In the $\mathcal{A}$-space of degree pq, the following relations are satisfied for all $v, w \in S^{2n+1}$, $\lambda \in S^1$:*

1. $[(\lambda v, d) \circ (w, e)] = [(v, d) \circ (w, e)]$ *and* $[(v, d) \circ (\lambda w, e)] = [(A_w^{\lambda^{e-1}} v, d) \circ (w, e)]$.
2. $[(v, d) \circ (w, e)] = [(w, e) \circ (v, d)]$ *if either $v \parallel w$ or $v \perp w$.*

Furthermore, these are the only *relations in $\mathcal{A}(n)_{pq}$.*

Proof of Theorems 3.4.11 and 3.4.12. The proof has virtually the same structure as the proof for Theorems 3.4.5 and 3.4.6, so we will not repeat ourselves. The only place we must take care is in extracting the order of the prime degrees of the atomic polynomial maps. However, because there are only two such maps with distinct degrees, this can be done by inspecting the form of the hypersurface Γ we defined above. ☐

In the general case, we have the formula

$$\begin{aligned} Z[v_1 \circ \cdots \circ v_k] &= V(\det(Dv_k)_z) \cup V(\det(Dv_{k-1})_{v_k(z)}) \cup \ldots \\ &\qquad \cdots \cup V(\det(Dv_1)_{v_2 \circ \cdots \circ v_k(z)}) \\ &= V(\langle z, v_k \rangle) \cup V(\langle v_k(z), v_{k-1} \rangle) \cup \cdots \\ &\qquad \cdots \cup V(\langle v_2 \circ \cdots \circ v_k(z), v_1 \rangle) \\ &= (\mathbf{C}v_k)^\perp \cup v_k^{-1}((\mathbf{C}v_{k-1})^\perp) \cup v_k^{-1} v_{k-1}^{-1}((\mathbf{C}v_{k-2})^\perp) \cup \cdots \\ &\qquad \cdots \cup v_k^{-1} v_{k-1}^{-1} \cdots v_2^{-1}((\mathbf{C}v_1)^\perp) \end{aligned}$$

for the critical point set of an element of the $\mathcal{A}$-space. In this form, it is conceivable that the general cases of Conjecture 3.4.1 (Injectivity of Z) and Conjecture 3.4.2 () may be proven via an inductive argument. We choose not to embark on this endeavour within this thesis.

References

[Bro62] Edgar H. Brown. "Cohomology Theories". In: *Annals of Mathematics* 75.3 (1962), pp. 467–484. ISSN: 0003486X, 19398980.

[Mil65] John W. Milnor. *Topology from the Differentiable Viewpoint*. Princeton Landmarks in Mathematics and Physics. Princeton, NJ: Princeton University Press, 1965.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic Classes*. Vol. 76. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 1974.

[BM76] G. Brumfiel and I. Madsen. "Evaluation of the transfer and the universal surgery classes". In: *Inventiones mathematicae* 32.2 (1976), pp. 133–169.

[LW82] Anatoly S. Libgober and John W. Wood. "Differentiable structures on complete intersections—I". In: *Topology* 21.4 (1982), pp. 469–482.

[Hus94] Dale Husemöller. *Fibre Bundles*. 3rd ed. Graduate Texts in Mathematics. New York, NY: Springer-Verlag New York, 1994, pp. XX, 356.

[Hat01] Allen Hatcher. *Algebraic Topology*. Online Version. Cambridge University Press, 2001. ISBN: 978-0521795401. URL: https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*. 2nd ed. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2001.

[Hat17] Allen Hatcher. *Vector Bundles and K-Theory*. Version 2.2, November 2017. Available online at https://pi.math.cornell.edu/~hatcher/VBKT/VBKTpage.html, 2017.

[CG21] Diarmuid Crowley and Xing Gu. *On $H^*(BPU_n;\mathbb{Z})$ and Weyl group invariants*. 2021. arXiv: 2103.03523 [math.AT]. URL: https://arxiv.org/abs/2103.03523.

[CN23] D. Crowley and Cs. Nagy. *The smooth classification of 4-dimensional complete intersections*. To appear. 2023.