Split polynomials and the Sullivan Conjecture

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These slides have been revised in response to examiner feedback following the initial presentation.

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\begin{presentation}
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Once upon a time...

Once upon a time ... there were the hypersurfaces

Definition

Let $f \in \mathbf{C}[T_0, \dots, T_{n+1}]$ be a homogeneous polynomial of degree $d > 0$. Assume 0 is a regular value of f. Then

$$X_n(d) = X_n(f) := \{\, [z] \in \mathbf{C}P^{n+1} \mid f(z) = 0 \,\}$$

is called a hypersurface.

- It is a complex manifold with dimension n and codimension 1.
- We mainly focus on the underlying orientable smooth manifold.
- The number d is called the degree.

Example

- $X_1(d) \subseteq \mathbf{C}P^2$ is a closed orientable surface.
- $X_2(T_0^4 + T_1^4 + T_2^4 + T_3^4)$ is a K3 surface.

Out of the hypersurfaces, the complete intersections arose

Definition

Let $f_1, \dots, f_k \in \mathbf{C}[T_0, \dots, T_{n+k}]$ be homogeneous polynomials of degree $d_1, \dots, d_k > 0$. Assume 0 is a regular value of each f_i. When the k hypersurfaces

$$X_n(\underline{d}) = X_n(f_1, \dots, f_k) \coloneqq X_{n+k-1}(f_1) \cap \dots \cap X_{n+k-1}(f_k) \subseteq \mathbf{C}P^{n+k}$$

intersect *transversely*, their intersection is called a complete intersection.

- It is a complex manifold with dimension n and codimension k.
- We call $\underline{d} = \{d_1, \dots, d_k\}$ (a *multi*set) the multidegree.
- The product $d = d_1 \cdots d_k$ is called total degree.

Example

Let $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$. The nilpotent cone of $\mathfrak{gl}_3$ is $X_5(\mathrm{tr}(T), \mathrm{tr}(T^2) - \mathrm{tr}(T)^2, \det(T))$.

By the degree-genus formula's hand, the hypersurfaces fell

What do we know about complete intersections as smooth manifolds?

By a result due to Thom [CN23, §2.1], the diffeomorphism type of a complete intersection depends only on the multidegree $\underline{d} = \{d_1, \dots, d_k\}$, not on the polynomials.

Example (The degree-genus formula for surfaces)

The hypersurface $X_1(d) \subseteq \mathbf{C}P^2$ is closed orientable surface. By the classification of closed surfaces, it is diffeomorphic to the genus g surface F_g for some g. The degree-genus formula says that

$$g = \frac{(d-1)(d-2)}{2}.$$

It's always an integer!

Complete intersections of dimension 1 too were conquered

Generalisation to multidegrees

In general, $X_1(d_1, \dots, d_k) \subseteq \mathbf{C}P^{1+k}$ is a closed orientable surface. It is diffeomorphic to F_g for some g. The genus is given by the formula

$$g = \frac{2 - d_1 \cdots d_k(k + 2 - (d_1 + \cdots + d_k))}{2}.$$

Yes, this is also always an integer!

Example

We can find collections of integers whose sum and product are the same:

- $\{d_1, d_2, d_3\} = \{6, 6, 1\}$: $6 + 6 + 1 = 13$, $6 \cdot 6 \cdot 1 = 36$.
- $\{d_1, d_2, d_3\} = \{2, 2, 9\}$: $2 + 2 + 9 = 13$, $2 \cdot 2 \cdot 9 = 36$.
- Therefore $X_1(6, 6, 1) \approx X_1(2, 2, 9) \approx F_{145}$.

But yet, for higher dimensions, they held mysteries to be uncovered

Conjecture

The Sullivan Conjecture states that for $n \geqslant 3$, two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic if they have the same Sullivan data:

1. *the total degree $d = d_1 \cdots d_k$;*
2. *the Pontryagin classes regarded as integers(!); and*
3. *the Euler characteristic.*

For a fixed n, the above integers are all *polynomials* in the degrees $d_1, \dots, d_k$.

Example

The Sullivan Conjecture holds for $n = 4$ due to [CN23]. For example,

$$X_4(\underbrace{3, \dots, 3}_{150}, \underbrace{7, \dots, 7}_{89}, \underbrace{9, \dots, 9}_{65}, 15, \underbrace{25, \dots, 25}_{130}) \quad \text{and} \quad X_4(\underbrace{5, \dots, 5}_{261}, \underbrace{21, \dots, 21}_{89}, \underbrace{27, \dots, 27}_{64})$$

are diffeomorphic.

The road ahead

Setting the scene
- Complete intersections and the Sullivan conjecture
- Introducing fibrewise degree-d maps

Introducing the main characters: split polynomials
- Introducing the $\mathcal{A}$-space

On the topic of classifying spaces
- Classifying fibrewise *split polynomial* maps
- Vector bundles over the $\mathcal{A}$-space

Cohomology of the $\mathcal{A}$-space

Fibrewise degree-d maps

How do we study complete intersections?

Complete intersections arise in another way.

- Let γ denote the conjugate of the tautological bundle over $\mathbf{C}P^n$.
- Let $f_{\underline{d}} : \gamma \oplus \cdots \oplus \gamma \to \gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}$ denote the tautological map.

If we restrict $f_{\underline{d}}$ to the disc bundle $D(\gamma \oplus \cdots \oplus \gamma)$, then $X_n(\underline{d})$ arises as the transverse intersection of $f_{\underline{d}}$ with the zero section for certain choices of homotopy.

$$\begin{tikzcd}
D(\gamma \oplus \cdots \oplus \gamma) \arrow[r, "f_{\underline{d}}"] \arrow[d] & D(\gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}) \arrow[d] \\
\mathbf{C}P^n \arrow[r, equal] & \mathbf{C}P^n \arrow[u, bend right, "\text{zero section}"']
\end{tikzcd}$$

We call $f_{\underline{d}}$ the normal map of $X_n(\underline{d})$.

Fibrewise degree-d maps

Definition

A fibrewise degree-d map is a fibre preserving map $f : S(E^n) \to S(F^n)$ between the sphere bundles of two (complex) vector bundles which is degree d on each fibre.

$$\begin{tikzcd} S(E^n) \arrow[rr, "f"] \arrow[dr] & & S(F^n) \arrow[dl] \\ & X & \end{tikzcd}$$

We define a functor

$$\mathcal{F}_d : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Sets}, \quad \mathcal{F}_d(X) \coloneqq \{ f : S(E) \to S(F) \}/\text{stabilisation \& homotopy},$$

giving the stable homotopy classes of fibrewise degree-d maps over a space X.

Example

The normal map $f_{\underline{d}} : S(\gamma \oplus \cdots \oplus \gamma) \to S(\gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k})$ is a fibrewise degree-d map.

All of the fibrewise degree-d maps

How do we classify fibrewise degree-d maps?

The functor $\mathcal{F}_d$ is representable, due to Brown [Bro62], by a classifying space which we denote by $(QS^0/U)_d$. In other words, there is a natural bijection

$$\mathcal{F}_d(X) \approx [X, (QS^0/U)_d].$$

Remark (About the notation)

The spaces QS^0 and U are the direct limits

$$QS^0 := \varinjlim_n \mathrm{Map}(S^n, S^n) \qquad \text{and} \qquad U := \varinjlim_n U(n)$$

under the standard inclusions.

(Their appearance in the notation $(QS^0/U)_d$ is following the work of Brumfiel and Madsen [BM76], and does not hold any precise mathematical meaning.)

But why fibrewise degree-d maps?

Recall the normal map $f_{\underline{d}} : \gamma \oplus \cdots \oplus \gamma \to \gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}$ over $\mathbf{C}P^n$ from which arises a complete intersection $X_n(\underline{d})$. On the sphere bundles, this is a fibrewise degree-d map, and therefore it has a classifying map $c_{\underline{d}} : \mathbf{C}P^n \to (QS^0/U)_d$, called the normal invariant.

$$\begin{tikzcd}
(S(\gamma \oplus \cdots \oplus \gamma) \arrow[r, "f_{\underline{d}}"] & S(\gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (S(E^{\mathrm{univ}}) \arrow[r, "f^{\mathrm{univ}}"] & S(F^{\mathrm{univ}})) \arrow[d] \\
& \mathbf{C}P^n \arrow[rr, "c_{\underline{d}}"] & & (QS^0/U)_d
\end{tikzcd}$$

Theorem (Crowley and Nagy [CN23, Theorem 5.17])

Let $n \geqslant 3$. The normal invariants $c_{\underline{d}}$ and $c_{\underline{d}'}$ for complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are homotopic if and only if $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic.

All of the fibrewise degree-d maps, again

So which space is $(QS^0/U)_d$?

A priori, we do not know what the space $(QS^0/U)_d$ is. In my thesis, I construct a model for this classifying space.

Let QS^0_d denote the degree-d component of QS^0. It is equipped with a left U-action by pre-composition.

Theorem A (F. '24, A model for $(QS^0/U)_d$)

The homotopy quotient $QS^0_d /\!\!/ U$, defined as the balanced product

$$QS^0_d /\!\!/ U \coloneqq EU \underset{U}{\times} QS^0_d,$$

is a model for the classifying space of fibrewise degree-d maps.

Fibrewise degree-d maps: too much?

Are fibrewise degree-d maps what we want?

Recall the normal map $f_{\underline{d}} : \gamma \oplus \cdots \oplus \gamma \to \gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}$.

Let us simplify and consider when $k = 1$. We can describe $f_d : \gamma \to \gamma^{\otimes d}$ on each fibre just the dth power map

$$\lambda v \longmapsto \lambda v \otimes \cdots \otimes \lambda v = \lambda^d v \otimes \cdots \otimes v.$$

This is not an arbitrary continuous map, but rather a polynomial.

So we are looking to classify fibrewise *polynomial* maps, not fibrewise maps in general.

Introducing the main characters

Split polynomials

What is a split polynomial?

Definition

The prototypical split polynomial is the dth power map $z \mapsto z^d$.

In the normal map $f_{\underline{d}} : \gamma \oplus \cdots \oplus \gamma \to \gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}$, this occurs on each of the 1-dimensional subspaces belonging to a copy of γ.

We can replicate this on $\mathbf{C}^{n+1}$ in the following way:

1. Pick a direction $v \in S^{2n+1}$, and extend v to an ordered orthonormal basis $\beta(v) = (v, b_1, \ldots, b_n)$ of $\mathbf{C}^{n+1}$.
2. Express the elements of $\mathbf{C}^{n+1}$ using coordinates with respect to this basis:

$$\begin{pmatrix} z_0 & z_1 & \cdots & z_n \end{pmatrix}_{\beta(v)} := z_0 v + z_1 b_1 + \cdots + z_n b_n.$$

3. An *atomic* split polynomial (v, d) is the dth power map in v-direction:

$$(v, d) \cdot \begin{pmatrix} z_0 & z_1 & \cdots & z_n \end{pmatrix}_{\beta(v)} = \begin{pmatrix} z_0^d & z_1 & \cdots & z_n \end{pmatrix}_{\beta(v)}.$$

What is a split polynomial?

Definition

The split polynomial space is the monoid under composition generated by the atomic split polynomials (v, d), and unitary maps $A \in U(n+1)$. It is a topological submonoid of $\mathrm{Map}(\mathbf{C}^{n+1}, \mathbf{C}^{n+1})$.

We denote the split polynomial space by $SP(n)$.

Example (Normal form)

Because $A \circ (v, d) = (Av, d) \circ A$, a generic split polynomial has the form

$$f = A \circ (v_1, d_1) \circ \cdots \circ (v_k, d_k).$$

Introducing the $\mathcal{A}$-space

We can define a left $U(n+1)$-action on the split polynomials by composition on the left:

$$A \cdot f = A \circ f.$$

This action is free.

Definition

The $\mathcal{A}$-space is quotient of $SP(n)$ under the $U(n+1)$-action. We denote the $\mathcal{A}$-space by $\mathcal{A}(n)$.

Why quotient by the unitary action?

- Each split polynomial f has a set of critical points $Z[f]$ in the domain where its derivative Df is not surjective.
- Two split polynomials defining the same equivalence class in the $\mathcal{A}$-space have the same critical points.

Fantastic critical points and where to find them

These critical points $Z[f]$ are actually algebraic subsets of $\mathbf{C}^{n+1}$ formed by taking unions of hypersurfaces.

Explicitly, it is the union of the vanishing loci

$$
\begin{aligned}
&Z[(v_1, d_1) \circ \cdots \circ (v_k, d_k)] \\
&\quad = V(\langle z, v_k \rangle) \qquad \leftarrow \text{hyperplane!} \\
&\qquad \cup V(\langle (v_k, d_k) \cdot z, v_{k-1} \rangle) \\
&\qquad \cup V(\langle (v_{k-1}, d_{k-1}) \cdot (v_k, d_k) \cdot z, v_{k-2} \rangle) \\
&\qquad \cup \cdots \\
&\qquad \cup V(\langle (v_2, d_2) \cdots (v_{k-1}, d_{k-1}) \cdot (v_k, d_k) \cdot z, v_1 \rangle).
\end{aligned}
$$

Figure 1: The real points of $Z[f]$.

Structure of the split polynomials

By studying the critical point set, we can answer questions such as the following.

When do atomic split polynomials commute?

Given atomic split polynomials (v, d) and (w, e), the two different compositions

$$(v, d) \circ (w, e) = (w, e) \circ (v, d)$$

are equal if and only if $v \parallel w$ or $v \perp w$.

Decomposition by degree

The degree map

The map $\deg : SP(n) \to \mathbf{Z}$ is locally constant, and therefore defines a decomposition of the split polynomials by degree:

$$SP(n) = \bigsqcup_{d \in \mathbf{Z}} SP(n)_d, \qquad SP(n)_d \coloneqq \deg^{-1}(d).$$

This decomposition also carries over to the $\mathcal{A}$-space:

$$\mathcal{A}(n) = \bigsqcup_{d \in \mathbf{Z}} \mathcal{A}(n)_d, \qquad \mathcal{A}(n)_d \coloneqq SP(n)_d / U(n+1).$$

The degree-d components can be studied by looking at the prime factorisation of d.

Structure of the $\mathcal{A}$-space

The atomic $\mathcal{A}$-space

The subspace of $\mathcal{A}(n)_d$ consisting of atomic split polynomials is homeomorphic to $\mathbf{C}P^n$. For a prime degree p, the entire $\mathcal{A}(n)_p$ is atomic.

The degree-pq $\mathcal{A}$-space

When the degree is the product of two primes pq, maps in $\mathcal{A}(n)_{pq}$ can only consist of compositions of two atomic split polynomials.

- **When $p = q$:** $\mathcal{A}(n)_{p^2}$ can *roughly** be described as pairs $[v, w]$ subject to a relation $[v, w] = [w, v]$ if $v \perp w$.
- **When $p \neq q$:** $\mathcal{A}(n)_{pq}$ can *roughly** be described as pairs $[(v, p), (w, q)]$ or $[(v, q), (w, p)]$, with $[(v, p), (w, q)] = [(w, q), (v, p)]$ if $v \parallel w$ or $v \perp w$.

Commutativity occurs when the directions are parallel or perpendicular.

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*There are some additional relations

On the topic of classifying spaces

Stabilising

The classifying space $QS^0_d /\!\!/ U$ for the functor $\mathcal{F}_d$ classifies fibrewise degree-d maps up to *stabilisation*. So we would like to *stabilise* our split polynomials too.

Definition
Under the standard inclusions $\mathbf{C}^{n+1} \hookrightarrow \mathbf{C}^{n+2}$, we can take the direct limit

$$SP_d \coloneqq \varinjlim_n SP(n)_d, \qquad \text{and} \qquad \mathcal{A}_d \coloneqq \varinjlim_n \mathcal{A}(n)_d.$$

These are the stable split polynomial space and stable $\mathcal{A}$-space of degree d.

The free U-action
SP_d inherits a *free* U-action from each of the finite-dimensional subspaces. Therefore:

- The stable $\mathcal{A}$-space is also the quotient SP_d/U.
- We can also construct the *homotopy* quotient $SP_d /\!\!/ U$.

Fibrewise *split polynomial* maps

Recall the normal invariant $c_{\underline{d}} : \mathbf{C}P^n \to QS^0_{\underline{d}} /\!\!/ U$, which is the classifying map for a normal map $f_{\underline{d}} : \gamma \oplus \cdots \oplus \gamma \to \gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}$ of a complete intersection. From our construction of the model $QS^0_{\underline{d}} /\!\!/ U$, $c_{\underline{d}}$ *a fortiori* factors through $SP_{\underline{d}} /\!\!/ U$.

$$\begin{tikzcd} \mathbf{C}P^n \arrow[rr, "c_{\underline{d}}"] \arrow[dr, dashed, "c^{\mathcal{A}}_{\underline{d}}"'] & & QS^0_{\underline{d}} /\!\!/ U \\ & \mathcal{A}_{\underline{d}} \arrow[ur, hook] & \end{tikzcd}$$

In fact, because the U-action on $SP_{\underline{d}}$ is free, there is a fibration

$$EU \longrightarrow SP_{\underline{d}} /\!\!/ U \longrightarrow \mathcal{A}_{\underline{d}}$$

which yields a homotopy equivalence $SP_{\underline{d}} /\!\!/ U \simeq \mathcal{A}_{\underline{d}}$.

What can we hope for?

Conjecture (Crowley and Nagy)

If the normal invariants $c_{\underline{d}}, c_{\underline{d'}} : \mathbf{C}P^n \to QS^0_{\underline{d}} /\!\!/ U$ are homotopic*, then maps into the $\mathcal{A}$-space $c^{\mathcal{A}}_{\underline{d}}, c^{\mathcal{A}}_{\underline{d'}} : \mathbf{C}P^n \to \mathcal{A}_{\underline{d}}$ are* already homotopic*.*

$$\begin{tikzcd}
\mathbf{C}P^n \arrow[rr, shift left, "c_{\underline{d}}"] \arrow[rr, shift right, "c_{\underline{d'}}"'] \arrow[dr, shift left, "c^{\mathcal{A}}_{\underline{d}}"] \arrow[dr, shift right, "c^{\mathcal{A}}_{\underline{d'}}"'] & & QS^0_{\underline{d}} /\!\!/ U \\
& \mathcal{A}_{\underline{d}} \arrow[ur, hook] &
\end{tikzcd}$$

Vector bundles over the $\mathcal{A}$-space

Because the $\mathcal{A}$-space is a quotient of $SP(n)$ by a free $U(n+1)$-action, the quotient map $SP(n) \to \mathcal{A}(n)$ becomes a principal $U(n+1)$-bundle. We get for free an associated complex vector bundle $V(SP(n)) \to \mathcal{A}(n)$.

Definition

Let $d = p_1 \cdots p_k$ be the prime factorisation of d. The maximal anti-diagonal Δ^-_d of $\mathcal{A}(n)_d$ is the subspace consisting of products

$$[f] = [(v_1, p_1) \circ \cdots \circ (v_k, p_k)], \qquad \text{where} \qquad v_i \perp v_j \text{ for all } i \neq j.$$

This is the subspace where the atomic split polynomials are maximally commutative.

Why anti-diagonals?

Observe that normal maps $f_{\underline{d}} : \gamma \oplus \cdots \oplus \gamma \to \gamma^{\otimes d_1} \oplus \cdots \oplus \gamma^{\otimes d_k}$ on each fibre belong to the (possibly not maximal) anti-diagonal.

Vector bundles over the anti-diagonal

Assume that $d = p_1 \cdots p_k$ is a product of distinct primes.

Then the maximal anti-diagonal is diffeomorphic to the flag manifold $Fl_k(\mathbf{C}^{n+1})$, consisting of k orthogonal lines labelled by the primes $p_1, \ldots, p_k$. So there are k tautological line bundles

$$\kappa_{p_i} = \{ ([\cdots \circ (v_i, p_i) \circ \cdots], w) \mid w \in \mathbf{C}v_i \} \longrightarrow \Delta_d^-,$$

where at each point the fibre is the line corresponding to the prime p_i.

Theorem B (F. '24)
The vector bundle $V(SP(n)) \to \mathcal{A}(n)$ restricted to the maximal anti-diagonal is

$$\kappa_{p_1}^{\otimes p_1} \oplus \cdots \oplus \kappa_{p_k}^{\otimes p_k} \oplus (\kappa_{p_1} \oplus \cdots \oplus \kappa_{p_k})^{\perp} \longrightarrow \Delta_d^- \approx Fl_k(\mathbf{C}^{n+1})$$

A little bit of cohomology

Cohomology of $\mathcal{A}_{p^2}$, p prime

In my thesis, I have also calculated the cohomology groups of the $\mathcal{A}$-space in various degrees.

Theorem C (F. '24)

The integral cohomology groups of $\mathcal{A}_{p^2}$ are given by

$$H^i(\mathcal{A}_{p^2};\mathbf{Z}) \approx \begin{cases} \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus j}, & \text{for } i = 4j, \\ \mathbf{Z}^{\oplus(j+1)} \oplus \mathbf{Z}_2^{\oplus(j+1)}, & \text{for } i = 4j+2, \\ 0, & \text{otherwise.} \end{cases}$$

Cohomology of $\mathcal{A}_{pq}$, p and q distinct primes

Theorem D (F. '24)
The integral cohomology groups of $\mathcal{A}_{pq}$ in dimensions 0, 1, and 2 are given by

$$H^i(\mathcal{A}_{pq};\mathbf{Z}) \approx \begin{cases} \mathbf{Z}, & \text{for } i = 0, \\ \mathbf{Z}, & \text{for } i = 1, \\ \mathbf{Z}_{(p-1,q-1)}, & \text{for } i = 2, \end{cases}$$

where $(p-1, q-1)$ denotes the greatest common divisor of $p-1$ and $q-1$.

The rational cohomology groups of $\mathcal{A}_{pq}$ are given by

$$H^i(\mathcal{A}_{pq};\mathbf{Q}) \approx \begin{cases} \mathbf{Q}, & \text{for } i = 0, 1, \\ \mathbf{Q}^{\oplus(j-1)}, & \text{for } i = 2j, \text{ where } j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

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References

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